Withholding Verifiable Information
Supplementary Appendix (For Online Publication)

Kun Zhang*

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Contents

*Arizona State University, Email: kunzhang@asu.edu.

B Multidimensional State Space

Consider the following multidimensional extension of the baseline model: The state space is $\Omega = [0,1]^m$ with generic element $\boldsymbol{\omega} = (\omega_1, \omega_2, \dots, \omega_m)$. The state is distributed according to an absolutely continuous CDF F whose density f is strictly positive. For each $j = 1, \dots, m$, let x_j denote the posterior mean of the j-th element of the state; call $\boldsymbol{x} = (x_1, x_2, \dots, x_m)$ posterior mean of the state; clearly, $\boldsymbol{x} \in [0,1]^m$. Let $A_0, A_1, \dots, A_{n-1}$ be closed sets such that action a_i is optimal for Receiver if and only if the posterior mean of the state $\boldsymbol{x} \in A_i$, and

(1) $\mu_F(A_i) > 0$ for all $i \in \{0, 1, \dots, n-1\}$;
(2) $\cup_{i=0}^{n-1} A_i = [0,1]^m$;
(3) for any $i, j \in \{0, 1, \dots, n-1\}$, $\mu_F(A_i \cap A_j) = 0$;
(4) for any $i, j \in \{0, 1, \dots, n-1\}$ with $i < j$, if $\boldsymbol{x} \in A_i$ and $\boldsymbol{y} \in A_j$, it cannot be that $\boldsymbol{x} > \boldsymbol{y}$.

The four conditions above naturally extend the assumption on Receiver's optimal actions in the baseline model to a multidimensional environment.

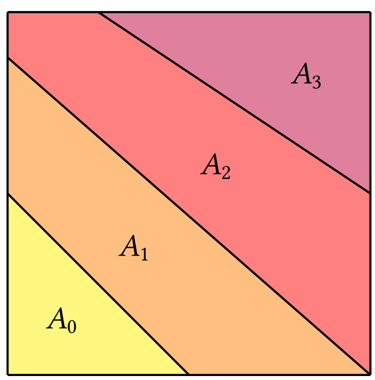

Figure 1: An illustration for Receiver's optimal actions when $m = 2$ and $|\mathcal{A}| = 4$.

Kleiner, Moldovanu, Strack, and Whitmeyer (2022) show that there exists an extreme point solution to the information design problem that partitions the state space into a finite collection of sets; and on each of them, at most $m+1$ signals can realize with positive probability, which is an analogy to Lemma 3.2. Furthermore, there exists a deterministic representation of the extreme point solution, which in turn yields Proposition B.1.

Proposition B.1. *An extreme point solution to the information design problem is implementable if and only if there exists an incentive compatible deterministic representation.*

To establish Proposition B.1, like in the proof of Proposition 3.4, I start from the following lemma.

Lemma B.2. *If an extreme-point solution is implementable, its outcome can be induced by an equilibrium in which Sender and Receiver only use deterministic messages and actions, respectively. Consequently, this equilibrium can be identified by a deterministic representation.*

Proof. The proof is analogous to the proof of Lemma A.1: the only difference is that a partitional element of the state space can be at most $m + 1$-pooling, and otherwise the same argument works. ∎

With the help of Lemma B.2, the proof of Lemma A.2 does not rely on the unidimensionality of the state, which yields the desired statement.

Although Receiver only takes one action, as shown in the following example, Proposition B.1 can be applied to some multi-action environments.

Example. The following example is taken from Dworczak and Martini (2019). Sender would like to have Receiver taking action on two dimensions, say 1 and 2, and she observes Receiver's payoffs $\boldsymbol{\omega} = (\omega_1, \omega_2)$ from taking actions 1 and 2, respectively. Receiver does not observe $\boldsymbol{\omega}$, and it is common knowledge that Receiver believes that $\boldsymbol{\omega}$ is distributed according to a symmetric distribution F with bounded, strictly positive density f. I further assume that actions are costly to Receiver, and hence for $j = 1, 2$, Receiver would like to take action j if and only if $\omega_j \geq \beta$, where $\beta \in (0, 1)$.[1]

Sender has state-independent preferences: she gets 0 if Receiver takes neither action, 1 if Receiver takes one action, and $2 + \alpha$ if Receiver takes both. The parameter $\alpha \in (-1, \infty)$ measures the complementarity between the two actions. To fit this example into my framework, I relabel "taking no action" as a_0, "taking one action" as a_1, and "taking both actions" as a_2. Consequently, $A_0 = [0, \beta]^2$, $A_1 = [0, \beta] \times [\beta, 1] \cup [\beta, 1] \times [0, \beta]$, and $A_2 = [\beta, 1]^2$; it is straightforward to check that conditions (1) to (4) above hold.

Proposition 6 in Dworczak and Martini (2019) identifies a commitment outcome of this problem; it can be shown that there exists $\bar{\alpha} \in \mathbb{R}$ such that this commitment outcome

[1] As indicated in Dworczak and Martini (2019), some further technical assumptions on β are needed; see page 2015-2016 in Dworczak and Martini (2019) for details.

is implementable if and only if $\alpha \geq \bar{\alpha}$. As pointed out by Dworczak and Martini (2019), Sender faces a tradeoff between increasing the probability of inducing both approvals versus the probability of inducing at least one approval. Consequently, when the complementarity between the two approvals is large enough, Sender is more willing to induce both approvals under commitment. In particular, in every state in which both actions are taken under full disclosure, the deterministic representation induced by the extreme point solution recommends both actions with probability 1. Therefore, incentive compatibility holds, which in turn implies that the extreme point solution is implementable.

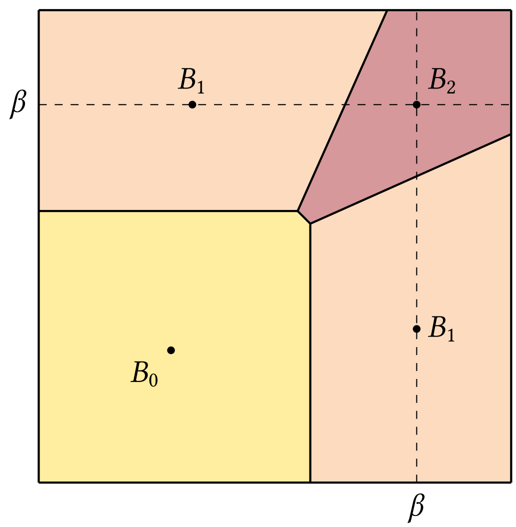

Figure 2: An illustration for the example. In this picture $\alpha = -0.5$ (and hence action 1 and 2 are "substitutes"), $\beta = 0.8$, and F is the bivariate uniform distribution. The maroon area, apricot area and yellow area are B_2, B_1 and B_0, respectively; $\{B_0, B_1, B_2\}$ is a deterministic representation of an extreme point solution. It is not difficult to see that incentive compatibility holds, and hence this solution is implementable.

An important consequence of Proposition B.1 is that even if the state space becomes multidimensional, Sender still does not benefit from commitment power when there are only two actions.

Corollary B.3. *If $|\mathcal{A}| = 2$, there exists an extreme point solution to the information design problem that can be implemented.*

Proof. With only two actions, (IC) reduces to $A_1 \subseteq B_1$. Suppose not, so $\text{int}A_1 \cap B_0 \neq \emptyset$. Because this set is of positive measure, by letting $\widetilde{B}_1 = B_1 \cup (A_1 \cap B_0)$ and $\widetilde{B}_0 = B_1 \setminus (A_1 \cap B_0)$, Receiver is still obedient and Sender's payoff is strictly improved because a_1 is played strictly more often. A contradiction. ■

C Remaining Omitted Proofs

C.1 Proof of Corollary 3.9

Recall that $\gamma_0 := 0$ and v_0 is normalized to zero; hence when $n = 3$, Equation (2) in the main text reduces to

$$\frac{v_2 - v_1}{\gamma_2 - \gamma_1} > \frac{v_1}{\min\{\gamma_1, \gamma_1 - h(\gamma_1; \gamma_2)\}} \tag{1}$$

And because $h(\gamma_1; \gamma_2) \geq 0$, the right-hand side of (1) further reduces to $v_1/(\gamma_1 - h(\gamma_1; \gamma_2))$. Then since f is increasing, $\gamma_2 - \gamma_1 \geq \gamma_1 - h(\gamma_1; \gamma_2)$; thus, if $v_2 - v_1 > v_1$, or $v_2 > 2v_1$, (1) must hold. Consequently, by Proposition 3.7, every bi-pooling solution can be implemented. By Lemma D.3, all bi-pooling solutions induce essentially the same canonical representation. Then because the set of bi-pooling solutions is the set of extreme points of the solution correspondence of the information design problem. Consequently, every solution must induce essentially the same canonical representation. Thus, every commitment outcome is implementable.

C.2 Proofs of Technical Lemmas for Proposition 4.2

C.2.1 Proof of Lemma A.9

Without loss of generality, let $\text{supp}(G_S) = \{\gamma_S, \gamma_{(1)}, \dots, \gamma_{(k-1)}\}$, where $\gamma_S \notin \{\gamma_i\}_{i=1}^{n-1}$, $k \leq n$, and $\gamma_S < \gamma_{(1)} < \gamma_{(2)} < \dots < \gamma_{(k-1)}$. Because $F(0) = G(0) = 0$, F is strictly increasing (since $f > 0$), and G_S is a step function, there exists $\varepsilon > 0$ such that $F(x) > G(x)$ for all $x \in [0, \varepsilon]$. By definition, $\int_0^1 G_S(q)\,\mathrm{d}q = \int_0^1 F(q)\,\mathrm{d}q$, and hence the set

$$D_1 = \left\{ x \in [\varepsilon, 1] \,:\, \int_0^x G_S(q)\,\mathrm{d}q = \int_0^x F(q)\,\mathrm{d}q \right\}$$

is nonempty. Let $d_1 = \inf D_1$. If $d_1 = 1$, set $m = 1$ and the proof is complete; suppose $d_1 < 1$.

Observe that $\gamma_S < d_1 < \gamma_{(k-1)}$: if $d_1 \leq \gamma_S$ then $F(x) > G(x) = 0$ for all $x \in [0, d_1)$, contradicting the definition of d_1; and if $d_1 \geq \gamma_{(k-1)}$, $F(x) < G_S(x) = 1$ for all $x \in [d_1, 1]$, and hence it cannot be that $\int_0^1 G_S(q)\,\mathrm{d}q = \int_0^1 F(q)\,\mathrm{d}q$, again a contradiction. Next I claim that $F(d_1) \geq G_S(d_1)$. Suppose to the contrary that $F(d_1) < G_S(d_1)$, then because G_S is a CDF and hence right-continuous, there must exist $\delta > 0$ such that $F(x) < G_S(x) = G_S(d_1)$ for all $x \in [d_1, d_1 + \delta]$, where the equality follows from the fact that G_S is a step function.

Then

$$\int_0^{d_1+\delta} F(q)\,\mathrm{d}q = \int_0^{d_1} F(q)\,\mathrm{d}q + \int_{d_2}^{d_1+\delta} F(q)\,\mathrm{d}q = \int_0^{d_1} G_S(q)\,\mathrm{d}q + \int_{d_2}^{d_1+\delta} F(q)\,\mathrm{d}q$$
$$< \int_0^{d_1} G_S(q)\,\mathrm{d}q + \int_{d_1}^{d_1+\delta} G_S(q)\,\mathrm{d}q,$$

which contradicts the assumption that G_S is a MPC of F.

Given these, let $j = \min\{i : \gamma_{(i)} > d_1\}$. Because $d_1 < \gamma_{(j)}$, $F(d_1) \geq G_S(d_1)$, F is strictly increasing, and G_S is a step function, there exists $\eta > 0$ such that $F(x) > G_S(x)$ for all $x \in [d_1, d_1+\eta]$. Now $\int_0^1 G_S(q)\,\mathrm{d}q = \int_0^1 F(q)\,\mathrm{d}q$ implies that the set

$$D_2 = \left\{ x \in [d_1+\eta, 1] : \int_0^x G_S(q)\,\mathrm{d}q = \int_0^x F(q)\,\mathrm{d}q \right\}$$

is nonempty. Let $d_2 = \inf D_2$. If $d_2 = 1$, set $m = 2$ and the proof is complete; suppose $d_2 < 1$. Similar to the previous paragraph, one can show that $d_2 > \gamma_{(j)}$ and $F(d_2) \geq G_S(d_2)$. Proceed inductively, eventually one can find d_t with $t \leq k \leq n$ such that $d_t > \gamma_{(k-1)}$. It must be that $d_t = 1$: suppose not, then because F is strictly increasing, $F(x) < 1$ on $(d_k, 1)$; but $G_S(x) = 1$ on the same interval, which implies that $\int_0^1 G_S(q)\,\mathrm{d}q > \int_0^1 F(q)\,\mathrm{d}q$, a contradiction. Now set $m = t$, the proof of is complete.

C.2.2 Proof of Lemma A.10

By Lemma A.9, on each of the intervals such that the MPC constraint does not bind, the mass is redistributed to at most n points, and there can be at most n such intervals. I show that every such interval admits a laminar representation; and the definition readily implies that by taking union, the resulting representation is still laminar. Consequently, a laminar representation indeed exists. Denote an arbitrary interval on which the MPC constraint does not bind by I; that is, $\int_0^x G_S(q)\,\mathrm{d}q \leq \int_0^x F(q)\,\mathrm{d}q$ for all $q \in I$, and the inequality holds with equality only at a and b.

The proof is very similar to the proof of Lemma 11 in Candogan and Strack (2022), and hence I only provide an outline here; readers interested in details are directed to that paper. Let $K = |\operatorname{supp}(G_S) \cap I| \leq n-1$; the proof proceeds by induction on K. If $K = 1$, let $\operatorname{supp}(G_S) \cap I = \{\gamma_\ell\}$; then clearly $\{B_\ell\}$ where $B_\ell = I$ is a laminar representation of I. If $K = 2$, let $\operatorname{supp}(G_S) \cap I = \{\gamma_\ell, \gamma_m\}$ where $\ell < m$; then by Lemma 3.3, $\{B_\ell, B_m\}$ can be taken as a nested interval representation of I, which is clearly laminar.

Taking $K = 2$ as the base case, consider $K > 2$; the induction hypothesis holds for $K-1$. One can find a closed interval B_ℓ such that (i) $\mu_F(B_\ell) = g_S(\gamma_\ell)$, where g_S is the probability mass function (pmf) of G_S, and $\gamma_\ell = \min(\text{supp}\,(G_S) \cap I)$,[2] and (ii) $\mathbb{E}[\omega \mid \omega \in B_\ell] = \gamma_\ell$. Consequently, conditional on $\omega \notin B_\ell$, G_S only has $K-1$ mass points, and Lemma 12 in Candogan and Strack (2022) shows that it is a MPC of F. Now by invoking the inductive hypothesis, a laminar representation of I, $\{\hat{B}_i\}_{i\in\mathcal{T}}$, is obtained, where $\mathcal{T} := \{k \neq \ell : \gamma_k \in \text{supp}\,(G_S) \cap I\}$. For every $i \in \mathcal{T}$, let $B_i = \hat{B}_i \backslash B_\ell$. Since $\{\hat{B}_i\}_{i\in\mathcal{T}}$ is laminar, and $\ell < i$ for all $i \in \mathcal{T}$, $\{B_i\}_{i\in\mathcal{T}\cup\{\ell\}}$ is also laminar.

C.2.3 Proof of Lemma A.13

By Lemma A.7, it suffices to show that Sender's preferred ORE exists. The problem of finding a Sender's preferred ORE can be written as

$$
\begin{aligned}
\max_{\{B_i\}_{i=0}^{n-1}} \quad & \sum_{i=0}^{n-1} v_i \mu_F(B_i) \\
\text{s.t.} \quad & \cup_{i=0}^{n-1} B_i = [0,1] \\
& \mu_F(B_i \cap B_j) = \emptyset \text{ for all } i \neq j \\
& \gamma_i \leq \mathbb{E}[\omega \mid \omega \in B_i] \leq \gamma_{i+1} \text{ for all } B_i \text{ such that } \mu_F(B_i) > 0 \\
& A_i \subseteq \cup_{i\geq k} B_k
\end{aligned}
$$

Although an action j may be never recommended, one can nonetheless assume that B_j is nonempty by setting $\mu_F(B_j) = 0$. By Lemma A.11 and Lemma A.12, it is without loss of generality to assume that B_i is the union of at most $n-1$ intervals. Consequently, one can always set B_i as the union of exactly $n-1$ convex sets $\{B_{i,s}\}_{s=1}^{n-1}$ such that $\mu_F(B_{i,s'} \cap B_{i,s''}) = 0$ for all $s' \neq s''$: if action a_i is recommended with probability zero, B_i can be set as the union of $n-1$ singletons.

Let $\mathcal{C}_c([0,1])$ denote the set of closed, nonempty, and convex subsets of $[0,1]$ endowed with the Hausdorff distance; to simplify notation, I write $\mathcal{C}_c$ henceforth. By Proposition 1 in Ely (2019), $\mathcal{C}_c$ is compact; and by Tychonoff's theorem, $\mathcal{C}_c^{n(n-1)}$ is compact in the product

[2] The only difference between my proof and Candogan and Strack (2022)'s is that they choose the mass point of G_S that has the largest index number on I, and for my purpose I work with the smallest. Their proof; however, goes through despite this difference.

topology. The problem above can be transformed to

$$\max_{\{B_{i,s}\}\in\mathcal{C}_c^{n(n-1)}} \sum_{i=0}^{n-1} v_i \sum_{s=1}^{n-1} \mu_F(B_{i,s}) \tag{2}$$

$$\text{s.t.} \quad \cup_{i,s} B_{i,s} = [0,1]$$

$$\mu_F(B_{i',s'} \cap B_{i'',s''}) = 0 \text{ for all } (i',s') \neq (i'',s'')$$

$$\mu_F\left(\cup_{s=1}^{n-1} B_{i,s}\right)\gamma_i \leq \int_{\cup_{s=1}^{n-1} B_{i,s}} \omega \, \mathrm{d}\mu_F(\omega) \leq \mu_F\left(\cup_{s=1}^{n-1} B_{i+1,s}\right)\gamma_{i+1}$$

$$A_i \subseteq \cup_{i\geq k} \cup_{s=1}^{n-1} B_{k,s}$$

where the third constraint is equivalent to the conditional mean condition.

Define

$$\mathcal{D} = \left\{\{B_{i,s}\} \in \mathcal{C}_c^{n(n-1)} \; : \; \cup_{i,s} B_{i,s} = [0,1], \text{ and } \mu_F(B_{i',s'} \cap B_{i'',s''}) = 0 \text{ for all } (i',s') \neq (i'',s'')\right\};$$

I claim that $\mathcal{D}$ is compact. To show this, it is enough to show that $\mathcal{D}$ is a closed subset of $\mathcal{C}_c^{n(n-1)}$. Take any $\{B_{i,s}^m\}$ that converges to $\{B_{i,s}\}$ in the product topology, then $B_{i,s}^m \to B_{i,s}$ for each i and s. Consequently, because the limit of convergence in Hausdorff distance is preserved under unions,[3] $\cup_{i,s} B_{i,s}^m \to \cup_{i,s} B_{i,s}$. Therefore, if $\cup_{i,s} B_{i,s}^m = [0,1]$, it must be that $\cup_{i,s} B_{i,s} = [0,1]$. Furthermore, if $\mu_F(B_{i',s'}^m \cap B_{i'',s''}^m) = 0$ for all m and $(i',s') \neq (i'',s'')$, the same argument as the second paragraph in the proof of Lemma 2 in Ely (2019) shows that $\mu_F(B_{i',s'} \cap B_{i'',s''}) = 0$ for all $(i',s') \neq (i'',s'')$. Therefore, if $\{B_{i,s}^m\} \in \mathcal{D}$ for each m and $\{B_{i,s}^m\} \to \{B_{i,s}\}$, it must be that $\{B_{i,s}\} \in \mathcal{D}$. Thus, $\mathcal{D}$ is a closed subset of $\mathcal{C}_c^{n(n-1)}$.

Directly, problem (2) is equivalent to

$$\max_{\{B_{i,s}\}\in\mathcal{D}} \sum_{i=0}^{n-1} v_i \sum_{s=1}^{n-1} \mu_F(B_{i,s}) \tag{3}$$

$$\text{s.t.} \quad \left(\sum_{s=1}^{n-1} \mu_F(B_{i,s})\right)\gamma_i \leq \sum_{s=1}^{n-1} \int_{B_{i,s}} \omega \, \mathrm{d}\mu_F(\omega) \leq \left(\sum_{s=1}^{n-1} \mu_F(B_{i,s})\right)\gamma_{i+1} \tag{4}$$

$$A_i \subseteq \cup_{i\geq k} \cup_{s=1}^{n-1} B_{k,s} \tag{5}$$

where constraint (4) supersedes the the third constraint in problem (2) because for any $\{B_{i,s}\} \in \mathcal{D}$, $\mu_F(B_{i',s'} \cap B_{i'',s''}) = 0$ for all $(i',s') \neq (i'',s'')$.

[3] See, for example, Theorem 1.12.15 in Barnsley (2006).

By the extreme value theorem, to show that a solution to problem (3) exists, it suffices to show that (i) the objective function is continuous, and (ii) the constraint set is nonempty and compact. Clearly, the constraint set is nonempty: for each $i = 0, \dots, n-1$, consider

$$B_{i,1} = A_i, \text{ and } B_{i,s} = \{\gamma_{i+1}\} \text{ for all } s = 2, \dots, n-1,$$

then it is not hard to see that $\{B_{i,s}\}$ is feasible for this problem. Furthermore, by Proposition in Ely (2019), μ_F is continuous on $\mathcal{C}_c$, and hence the objective is continuous.

Since $\mathcal{D}$ is compact, to show that the constraint set is compact, it suffices to show that each of the constraints, (4) and (5), defines a closed subset of $\mathcal{D}$. Observe that if $C^m \to C$ and $D^m \to D$ with $C^m \subseteq D^m$ for all m, then $C \subseteq D$.[4] Then because the limit of convergence in Hausdorff distance is preserved under unions, if $A_i \subseteq \cup_{i \geq k} \cup_{s=1}^{n-1} B_{k,s}^m$ for each i, it must be that $A_i \subseteq \cup_{i \geq k} \cup_{s=1}^{n-1} B_{k,s}^m$ for each i. Hence, (5) defines a closed subset of $\mathcal{D}$.

Next I show that (4) does the same, which is equivalent to showing that if $\{B_{i,s}^m\} \to \{B_{i,s}\}$ where $\{B_{i,s}^m\} \in \mathcal{D}$ for each m, then

$$\left(\sum_{s=1}^{n-1} \mu_F(B_{i,s}^m) \right) \gamma_i \leq \sum_{s=1}^{n-1} \int_{B_{i,s}^m} \omega \, \mathrm{d}\mu_F(\omega) \leq \left(\sum_{s=1}^{n-1} \mu_F(B_{i,s}^m) \right) \gamma_{i+1}$$

for all m and i implies that

$$\left(\sum_{s=1}^{n-1} \mu_F(B_{i,s}) \right) \gamma_i \leq \sum_{s=1}^{n-1} \int_{B_{i,s}} \omega \, \mathrm{d}\mu_F(\omega) \leq \left(\sum_{s=1}^{n-1} \mu_F(B_{i,s}) \right) \gamma_{i+1}$$

for all i. Because μ_F is continuous on $\mathcal{C}_c$, $\sum_{s=1}^{n-1} \mu_F(B_{i,s}^m) \to \sum_{s=1}^{n-1} \mu_F(B_{i,s})$ for all i. Consequently, $\left(\sum_{s=1}^{n-1} \mu_F(B_{i,s}^m)\right) \gamma_i \to \left(\sum_{s=1}^{n-1} \mu_F(B_{i,s})\right) \gamma_i$, and $\left(\sum_{s=1}^{n-1} \mu_F(B_{i,s}^m)\right) \gamma_{i+1} \to \left(\sum_{s=1}^{n-1} \mu_F(B_{i,s})\right) \gamma_{i+1}$ for each i. Therefore, it only remains to show that

$$\sum_{s=1}^{n-1} \int_{B_{i,s}^m} \omega \, \mathrm{d}\mu_F(\omega) \to \sum_{s=1}^{n-1} \int_{B_{i,s}} \omega \, \mathrm{d}\mu_F(\omega),$$

which is a consequence of $\int_E \omega \, \mathrm{d}\mu_F(\omega)$ being continuous on $\mathcal{C}_c$. This fact is proved in Claim D.2.

[4]See, for example, page 15 in Ely (2019).

C.3 Proof of Claim 4.3

By Proposition 4.2, Sender's preferred equilibrium outcome can be represented by a laminar representation $\{B_0, B_1, B_2\}$. Claim D.1 shows that there exists $d \geq 0$ such that $B_0 = [0, d]$. Consequently, the laminar structure implies that either both B_1 and B_2 are intervals, or $\{B_1, B_2\}$ is a nested intervals representation. Either way, $\{B_0, B_1, B_2\}$ is a canonical representation.

C.4 Proof of Claim 4.4

To solve for the Sender's optimal equilibrium, I find a bi-pooling solution to the corresponding information design problem first, and check whether it is implementable using Proposition 3.5. If it is, the Sender's optimal equilibrium outcome coincides with the commitment outcome.[5]

Now suppose that no commitment outcome is implementable. By Claim 4.3, $\{B_1, B_2\}$ is a nested intervals representation; therefore, there exist y, h, and b such that $B_1 = [h, b]$, and $B_2 = [y, h] \cup [b, 1]$.[6] By Lemma D.5, there exist $y \geq 0$ and $h > 0$ such that

$$\mathbb{E}\left[\omega \mid \omega \in [h, b]\right] = \gamma_1 \quad \text{and} \quad \mathbb{E}\left[\omega \mid \omega \in [y, h] \cup [b, 1]\right] = \gamma_2.$$

Consequently, y and h can be implicitly defined as functions of b, and hence Sender's expected payoff in equilibrium can be parametrized by b, so long as $b \leq \gamma_2$:

$$\bar{V}(b) = v(a_1)[F(b) - F(h(b))] + v(a_2)[1 - F(b) + F(h(b)) - F(y(b))].$$

By Lemma D.6, $\bar{V}$ is increasing in b. Hence, the deterministic representation corresponding to Sender's preferred equilibrium can be found by setting $b = \gamma_2$, which yields the expression in the statement of the claim. This completes the proof.

[5] One may wonder what if the bi-pooling solution found above is not implementable, but there exists another bi-pooling solution that is implementable. This can never happen when there are three actions: by Lemma D.3, all bi-pooling solutions induce essentially the same canonical representation.

[6] If both B_1 and B_2 are intervals, just set $y = h$.

C.5 Proof of Claim 5.1

Let θ_q denote the cutoff quality that the buyer is indifferent between purchasing $q-1$ and q units: it solves

$$\theta_q U(q) - pq = \theta_q U(q-1) - p(q-1),$$

so $\theta_q = p/[U(q) - U(q-1)]$. Letting $\theta_0 = 0$ and $\theta_{n+1} = 1$, the buyer buys $q \in \{0, 1, \dots, n\}$ units of the product if and only if $\theta \in [\theta_q, \theta_{q+1}]$. If $P > 2A$, $U''(q)/[U'(q)]^2$ is strictly decreasing in q, and thus

$$\theta_{q+1} - \theta_q = \frac{p}{U(q+1) - U(q)} - \frac{p}{U(q) - U(q-1)}$$

is strictly decreasing in q. By noting that $v_q - v_{q-1} = p - c$ for all $q \in \{0, 1, \dots, n\}$, Claim 5.1 is implied by Corollary 3.8.

C.6 Proof of Claim 5.2

If either α_b^j increases for all $j = 1, \dots, N$, or β_b^j increases for all $j = 1, \dots, N$, or both, γ_2^j decreases for every j, and hence γ_j^m decreases. By Claim 4.4 when no commitment outcome can be implemented, Sender's payoff is given by

$$\bar{V}(\gamma_2^m) = v(a_1)[F(\gamma_2^m) - F(h(\gamma_2^m))] + v(a_2)[1 - F(\gamma_2^m) + F(h(\gamma_2^m)) - F(y(\gamma_2^m))],$$

where h and y are implicitly defined by

$$\mathbb{E}\left[\omega \mid \omega \in \left[h, \gamma_2^m\right]\right] = \gamma_1^m \quad \text{and} \quad \mathbb{E}\left[\omega \mid \omega \in [y, h] \cup \left[\gamma_2^m, 1\right]\right] = \gamma_2^m.$$

Now (to save notation, I write $v_i = v(a_i)$)

$$\bar{V}'(\gamma_2^m) = (v_1 - v_2)\left[f\left(\gamma_2^m\right) - f(h)\frac{\mathrm{d}h}{\mathrm{d}\gamma_2^m}\right] - v_2 f(y)\frac{\mathrm{d}y}{\mathrm{d}\gamma_2^m}. \tag{6}$$

By the implicit function theorem,

$$\frac{\mathrm{d}h}{\mathrm{d}\gamma_2^m} = -\frac{\left(\gamma_2^m - \gamma_1^m\right) f\left(\gamma_2^m\right)}{\left(\gamma_1^m - h\right) f(h)}, \tag{7}$$

$$\frac{\mathrm{d}y}{\mathrm{d}\gamma_2^m} = \frac{\left(\gamma_1^m - h\right)\left[1 - f\left(\gamma_2^m\right) + F(h) - F(y)\right]}{\left(\gamma_1^m - h\right)\left(\gamma_2^m - y\right) f(y)} - \frac{\left(\gamma_2^m - h\right)\left(\gamma_2^m - \gamma_1^m\right) f\left(\gamma_2^m\right)}{\left(\gamma_1^m - h\right)\left(\gamma_2^m - y\right) f(y)}. \tag{8}$$

Plug (7) and (8) into (6),

$$\bar{V}'(\gamma_2^m) = -\frac{\left(\gamma_1^m - h\right)\left[1 - f\left(\gamma_2^m\right) + F(h) - F(y)\right] v_2}{\left(\gamma_1^m - h\right)\left(\gamma_2^m - y\right)} - \frac{\left(\gamma_2^m - h\right)\left(\gamma_1^m - y\right) f\left(\gamma_2^m\right)}{\left(\gamma_1^m - h\right)\left(\gamma_2^m - y\right)} v_2 + \frac{\left(\gamma_2^m - h\right)\left(\gamma_2^m - y\right) f\left(\gamma_2^m\right)}{\left(\gamma_1^m - h\right)\left(\gamma_2^m - y\right)} v_1,$$

whose sign is determined by

$$-\left(\gamma_1^m - h\right)\left(1 - F\left(\gamma_1^m\right) + F(h) - F(y)\right) - \left(\gamma_2^m - h\right) f\left(\gamma_2^m\right)\left[\left(\gamma_1^m - y\right) v_2 - \left(\gamma_1^m - y\right) v_1\right]. \tag{9}$$

By Lemma D.5, if no comment outcome is implementable, it must be that $(\gamma_1^m - y)v_2 \leq (\gamma_2^m - y)v_1$. Consequently, the sign of the second term of (9) must be positive, and the first term has a strictly negative sign. Hence as γ_2^m decreases, the expert's payoff in her preferred equilibrium strictly decreases if the second term is larger in absolute value. This completes the proof.

C.7 Proof of Proposition 6.1

By Theorem 2 of Kleiner et al. (2021), for every bi-pooling solution, the state space is partitioned into a collection of intervals such that each of the intervals must satisfy one of the followings:

(i) it is a full disclosure interval in the sense that all states in the interval are perfectly revealed;

(ii) it is a pooling interval;

(iii) it is a bi-pooling interval.

Because a pooling interval can be interpreted as a special case of a bi-pooling interval, without loss of generality I can assume that every interval in the partition that does not feature full disclosure is a bi-pooling interval. I index these bi-pooling intervals by I; then the collection of bi-pooling intervals can be written as $\left([\underline{\omega}_i, \overline{\omega}_i]\right)_{i \in I}$. By Lemma 3.3, for each i there exists a bi-partition of $[\underline{\omega}_i, \overline{\omega}_i]$, denoted by $\{B_L^i, B_H^i\}$; finally, let $\mathcal{B} = \left(\{B_L^i, B_H^i\}\right)_{i \in I}$. For any closed subset m of $[0, 1]$, let $S_m = \arg\min_{x \in m} u(x)$; and for any $m \notin \mathcal{B}$, let $p(S_m \mid m) = 1$. Now Equation (7) (in the main text) implies that Sender never wants to deviate to full disclosure since such a deviation weakly decreases Sender's payoff. Then the belief system defined above guarantees that any other deviation is at most as good as full disclosure;

consequently, Sender never wants to deviate to any off-path message $m \notin \mathcal{B}$. Thus, the bi-pooling solution is implementable.

Now suppose that a bi-pooling solution G_B is implementable; then a similar argument as the proof of Lemma A.1 shows that a deterministic representation can induce its outcome. Suppose to the contrary that Equation (7) does not hold. Then for any deterministic representation of G_B, there exists say B^i_H, such that $u(z^i_H) < u(y) = \max_{x \in B^i_H} u(x)$. Since u is upper semicontinuous, there exists a nontrivial subinterval[7] of B^i_H such that for all states in this subinterval, full disclosure strictly dominates sending message B^i_H. This implies that; however, G_B is not implementable, a contradiction. This completes the proof.

C.8 Proof of Corollary 6.2

The "if" direction is straightforward because Equation (7) (in the main text) is stronger than Equation (6). For the other direction, if a bi-pooling solution G_B is implementable, by Proposition 6.1, for every bi-pooling interval $[\underline{\omega}_i, \overline{\omega}_i]$ with $\operatorname{supp}\left(G_B\big|_{[\underline{\omega}_i, \overline{\omega}_i]}\right) = \{z^i_L, z^i_H\}$ satisfies Equation (7). Because u is differentiable, the first-order condition implies that $u'(z^i_L) = u'(z^i_H) = 0$. Moreover, since $[\underline{\omega}_i, \overline{\omega}_i]$ is bi-pooled to $\{z^i_L, z^i_H\}$, regularity of u implies that there exists a DM-price function p such that it is affine on $[\underline{\omega}_i, \overline{\omega}_i]$ and is tangent to both $u(z^i_L)$ and $u(z^i_H)$. Therefore, the slope of p on $[\underline{\omega}_i, \overline{\omega}_i]$ must be zero, which implies that it is constant on that bi-pooling interval. Consequently, it must be that $u(z^i_L) = u(z^i_H) = 0$, and hence Equation (6) implies Equation (7).

C.9 Proof of Corollary 6.3

By Proposition 6.1, it suffices to focus on bi-pooling intervals. Then an argument analogous to the proof of Lemma A.3 establishes the result.

D Supplementary Results

D.1 Supplementary Results for the Proof of Proposition 4.2

Claim D.1. *Let $\{B_i\}_{i=0}^{n-1}$ be a laminar representation associated with a Sender's preferred ORE, and let $j = \min\{i : \operatorname{int}(B_i) \neq \emptyset\}$. If $\mathbb{E}[\omega \mid \omega \in B_j] > \gamma_j$, then $B_j = [0, d]$ for some $d < \gamma_{j+1}$.*

[7] That is, the Lebesgue measure of that interval is strictly positive.

Proof of Claim D.1. Because $\{B_i\}_{i=0}^{n-1}$ is a laminar representation, B_j must be an interval, and hence one can write $B_j = [\underline{b}_j, \bar{b}_j]$. Furthermore, it only suffices to show that $\text{int}(B_j) \cap \text{co}(B_k) = \emptyset$ for all $k > j$. Suppose not, so $B_j \subseteq \text{co}(B_k)$ for some $k > j$. The laminar structure implies that B_k is the union of at most n closed intervals; call these closed intervals its components. Let $[\alpha, \beta]$ and $[\zeta, \eta]$ be two components such that $\beta \leq \underline{b}_j$ and $\zeta \geq \bar{b}_j$; such two components can be found because $B_j \subseteq \text{co}(B_k)$. Because $\mathbb{E}[\omega \mid \omega \in B_k] = \gamma_k$ and $\mathbb{E}\left[\omega \mid \omega \in B_j\right] = \gamma_S > \gamma_j$, for small enough $\varepsilon > 0$, one can find $h(\varepsilon)$ such that

$$\mathbb{E}\left[\omega \middle| \omega \in \left(B_k \cup [\bar{b}_j - h(\varepsilon), \bar{b}_j]\right) \setminus (\beta - \varepsilon, \beta]\right] = \gamma_k, \quad \text{and} \quad \mathbb{E}\left[\omega \middle| \omega \in [\beta - \varepsilon, \beta] \cup [\bar{b}_j - h(\varepsilon), \bar{b}_j]\right] \geq \gamma_j.$$

Now define a new deterministic representation $\{\widetilde{B}_i\}_{i=0}^{n-1}$ by $\widetilde{B}_i = B_i$ if $i \neq j, k$, $\widetilde{B}_j = [\beta - \varepsilon, \beta] \cup [\bar{b}_j - h(\varepsilon), \bar{b}_j]$, and $\widetilde{B}_k = \left(B_k \cup [\bar{b}_j - h(\varepsilon), \bar{b}_j]\right) \setminus (\beta - \varepsilon, \beta]$. $\{\widetilde{B}_i\}_{i=0}^{n-1}$ is obedient by construction, and it is also incentive compatible because $\{B_i\}_{i=0}^{n-1}$ is; then by Lemma 4.1, there exists an ORE that is defined by it. Furthermore, it must be that $\mu_F(\widetilde{B}_k) > \mu_F(B_k)$: this is because $\mathbb{E}[\omega \mid \omega \in B_k] = \mathbb{E}\left[\omega \mid \omega \in \widetilde{B}_k\right] = \gamma_k$, but the unconditional mean of the former is strictly less than the latter by construction. Consequently, Sender's ex ante payoff is strictly higher in this new equilibrium since $j < k$, which contradicts the assumption that $\{B_i\}_{i=0}^{n-1}$ is a laminar representation defining a Sender's preferred equilibrium. This concludes the proof. ■

Claim D.2. *$\int_E \omega \, \mathrm{d}\mu_F(\omega)$ is continuous on $\mathcal{C}_c$.*

Proof of Claim D.2. I prove that $\int_E \omega \, \mathrm{d}\mu_F(\omega)$ is both upper- and lower-semicontinuous. To see that it is upper-semicontinuous, pick any $\varepsilon > 0$, and let $E \in \mathcal{C}_c$; I show that there exists $\delta > 0$ such that for every $E' \in N(E, \delta)$, where $N(E, \delta)$ is the δ-neighborhood of E, $\int_{E'} \omega \, \mathrm{d}\mu_F(\omega) < \int_E \omega \, \mathrm{d}\mu_F(\omega) + \varepsilon$. Because $E \in \mathcal{C}_c$, there exist $a, b \in [0, 1]$ such that $E = [a, b]$. The key observation here is that for any $E' \in N(E, \delta)$, it must be that $E' \subseteq [a - \delta, b + \delta]$. Then

$$\begin{aligned} \int_{a-\delta}^{b+\delta} \omega \, \mathrm{d}\mu_F(\omega) &= \int_{a-\delta}^{a} \omega \, \mathrm{d}\mu_F(\omega) + \int_{a}^{b} \omega \, \mathrm{d}\mu_F(\omega) + \int_{b}^{b+\delta} \omega \, \mathrm{d}\mu_F(\omega) \\ &\leq \mu_F([a - \delta, a]) + \int_E \omega \, \mathrm{d}\mu_F(\omega) + \mu_F([b, b + \delta]) \end{aligned}$$

where the inequality holds because $\omega \in [0, 1]$. For δ small enough, since μ_F is absolutely continuous with respect to the Lebesgue measure,

$$\int_{E'} \omega \, \mathrm{d}\mu_F(\omega) \leq \int_{a-\delta}^{b+\delta} \omega \, \mathrm{d}\mu_F(\omega) < \int_E \omega \, \mathrm{d}\mu_F(\omega) + \varepsilon.$$

Next I show that $\int_E \omega \, \mathrm{d}\mu_F(\omega)$ is lower semicontinuous, that is, there exists $\delta > 0$ such that for every $E' \in N(E, \delta)$, $\int_{E'} \omega \, \mathrm{d}\mu_F(\omega) > \int_{E'} \omega \, \mathrm{d}\mu_F(\omega) - \varepsilon$. Without loss of generality, assume $\delta < (b - a)/2$. Consequently, $[a + \delta, b - \delta]$ is an interval of positive measure, and for any $E' \in N(E, \delta)$, $[a + \delta, b - \delta] \subseteq E'$. Then

$$\begin{aligned}\int_{a+\delta}^{b-\delta} \omega \, \mathrm{d}\mu_F(\omega) &= \int_a^b \omega \, \mathrm{d}\mu_F(\omega) - \int_a^{a+\delta} \omega \, \mathrm{d}\mu_F(\omega) - \int_{b-\delta}^b \omega \, \mathrm{d}\mu_F(\omega) \\ &\geq \int_E \omega \, \mathrm{d}\mu_F(\omega) - a\mu_F([a - \delta, a]) - a\mu_F([b, b + \delta])\end{aligned}$$

where the inequality follows from the fact that $\omega \geq a$ on $[a, b]$. Consequently, $\int_E \omega \, \mathrm{d}\mu_F(\omega)$ is both upper- and lower-semicontinuous, and hence continuous. ■

D.2 Supplementary Results for Ternary Actions

In this subsection I assume that there are three actions, and hence $\mathcal{A} = \{a_0, a_1, a_2\}$. Recall that $\{\gamma_L, \gamma_H\}$ is feasible for the interval $[\underline{\omega}, \overline{\omega}]$ if there exists a mean preserving contraction of $F|_{[\underline{\omega}, \overline{\omega}]}$ whose support is $\{\gamma_L, \gamma_H\}$.

Claim D.3. *If there does not exist $y \in [0, \gamma_1]$ such that $\{\gamma_1, \gamma_2\}$ is feasible for $[y, 1]$, then every bi-pooling solution is implementable.*

Proof of Claim D.3. By Lemma A.6, $\{\gamma_1, \gamma_2\}$ is feasible for the interval $[y, 1]$ if and only if

(i) $y \leq \gamma_1 \leq m(y) \leq \gamma_2 \leq 1$, and

(ii) $\mathbb{E}[\omega \,|\, \omega \in [\eta(\gamma_1; y), 1]] \geq \gamma_2$,

where $m(y) = \mathbb{E}[\omega \,|\, \omega \in [y, 1]]$, and $\eta(\gamma_1; y)$ is such that $\mathbb{E}[\omega \,|\, \omega \in [y, \eta(\gamma_1)]] = \gamma_1$. Then if there does not exist $y \in [0, \gamma_1]$ such that $\{\gamma_1, \gamma_2\}$ is feasible for $[y, 1]$, there are two cases:

(a) (i) fails to hold for all $y \in [0, \gamma_1]$;

(b) (i) holds for some $y \in [0, \gamma_1]$, but (ii) fails for all such y's.

For Case (a), the only possibility is that $m(y) \geq \gamma_2$. If this is the case, it is clear that every solution G of the information design problem has $\text{supp}\,(G) \subseteq [\gamma_2, 1]$, and hence the essentially unique canonical representation has $B_2 = [0, 1]$, and incentive compatibility holds.

For Case (b), let me introduce some notation first. For each $i = 1, 2$, if there exists $h \geq 0$ such that $\mathbb{E}[\omega \,|\, \omega \in [h, 1]] = \gamma_i$, set $h(\gamma_i) = h$; otherwise let $h(\gamma_i) = 0$. Then for every $y \in [0, \gamma_1] \cap [h(\gamma_1), h(\gamma_2)]$, (i) holds. If (ii) fails for all such y's, it must be that $\mathbb{E}[\omega \,|\, \omega \in [\eta(\gamma_1; h(\gamma_1)), 1]] < \gamma_2$. Note that this is not possible if $h(\gamma_1) > 0$: if this is the case, $\eta(\gamma_1; h(\gamma_1)) = 1$ by definition, so it must be that $\mathbb{E}[\omega \,|\, \omega \in [\eta(\gamma_1; h(\gamma_1)), 1]] \geq \gamma_2$, a contradiction. Consequently, in Case (b), (ii) must fail for $y = 0$, and hence every solution G of the information design problem has $G(\gamma_2) - G(\gamma_2^-) = 1 - F(h(\gamma_2))$, and $\text{supp}\,(G) \subseteq [\gamma_1, \gamma_2]$. Thus, the essentially unique canonical representation of every G has $B_1 = [0, h(\gamma_2)]$ and $B_2 = [h(\gamma_2), 1]$. Since $h(\gamma_2) < \gamma_2$ by definition, incentive compatibility must hold. This completes the proof. ∎

Lemma D.4. *The corresponding canonical representations are essentially the same for all bi-pooling solutions to the information design problem.*[8]

Proof of Lemma D.4. It can be readily seen from the proof of Claim D.3 that, if there does not exist $y \in [0, \gamma_1]$ such that $\{\gamma_1, \gamma_2\}$ is feasible for $[y, 1]$, then all commitment outcomes generate essentially the same canonical representation. Now suppose there exists $y \in [0, \gamma_1]$ such that $\{\gamma_1, \gamma_2\}$ is feasible for $[y, 1]$. Let Y denote the set of such y's; Lemma A.6 implies that Y is a closed subset of $[0, \gamma_1]$. Then by Lemma 3.2, the commitment payoff can be identified by the lower endpoint of the bi-pooling interval. Hence, each of them corresponds to a point in Y that maximizes Sender's ex ante payoff[9]

$$\pi(y) = (1 - F(y)) \left[\frac{\gamma_2 - m(y)}{\gamma_2 - \gamma_1} v_1 + \frac{m(y) - \gamma_1}{\gamma_2 - \gamma_1} v_2 \right].$$

Taking derivative,

$$\pi'(y) = -f(y) \left[\frac{\gamma_2 - m(y)}{\gamma_2 - r_1} v_1 + \frac{m(y) - \gamma_1}{\gamma_2 - \gamma_1} v_2 \right] + (1 - F(y)) \frac{v_2 - v_1}{\gamma_2 - v_1} m'(y),$$

[8] By "essentially the same" I mean that two distinct canonical representations only differ on elements that are null sets.

[9] Recall that $v(a_0) = 0$; to simplify notation, I write v_i instead of $v(a_i)$.

where

$$m'(y) = \frac{(m(y) - y)f(y)}{1 - F(y)}.$$

Consequently,

$$\begin{aligned}\pi'(y) &= \frac{f(y)}{\gamma_2 - \gamma_1}\left[(m(y) - y)(v_2 - v_1) - (\gamma_2 - m(y))\, v_1 - (m(y) - \gamma_1)\, v_2\right], \\ &= \frac{f(y)}{\gamma_2 - \gamma_1}\left[V_2(\gamma_1 - y) - v_1(\gamma_2 - y)\right]\end{aligned}$$

and its sign is determined by the terms between the squared brackets, which implies that π is single-peaked in y. Therefore, there must exist a unique z that maximizes $\pi(y)$ on Y. As a consequence, every bi-pooling solution has essentially unique canonical representation $B_0 = [0, z]$, $B_1 = [\underline{b}_1, \bar{b}_1]$, and $B_2 = [z, \underline{b}_1] \cup [\bar{b}_1, 1]$. ∎

Lemma D.5. *If no commitment outcome is implementable, the Sender's preferred ORE is defined by* $\{B_0, B_1, B_2\}$ *such that* $B_0 = [0, y]$, $B_1 = [h, b]$, *and* $B_2 = [y, h] \cup [h, 1]$, *where* $b > \gamma_1$, *and* $h > 0$ *and* $y \geq 0$ *are defined by*

$$\mathbb{E}[\omega \mid \omega \in [h, b]] = \gamma_1 \quad \textit{and} \quad \mathbb{E}[\omega \mid \omega \in [y, h] \cup [b, 1]] = \gamma_2. \tag{10}$$

Furthermore,

$$v_2 \leq \frac{\gamma_2 - y}{\gamma_1 - y} v_1. \tag{11}$$

Proof of Lemma D.5. By Claim 4.3, the Sender's preferred ORE is defined by $\{B_0, B_1, B_2\}$ such that B_1 and B_2 form a pair of nested intervals; and by Proposition 4.2, if B_1 has nonempty interior, it must be that $\mathbb{E}[\omega \mid \omega \in B_1] = \gamma_1$, and $\mathbb{E}[\omega \mid \omega \in B_2] = \gamma_2$.[10] Then to obtain the statement it suffices to show three things: (1) $b > \gamma_1$, namely B_1 has nonempty interior; (2) there exists such an h; and (3) there exists such a y.

I show that B_1 has nonempty interior first. Suppose to the contrary that $\text{int}(B_1) = \emptyset$, then there are two cases: $B_2 = [0, 1]$, and $B_2 = [z, 1]$ for some $z > 0$. If $B_2 = [0, 1]$, then since Sender attains highest possible ex ante payoff in equilibrium, it must be that a commitment outcome is implementable, a contradiction. If instead $B_2 = [z, 1]$ for some $z > 0$, let $\bar{b}_0 := \sup B_0$. Then it must be that $\bar{b}_0$ solves $\mathbb{E}[\omega \mid \omega \in [\bar{b}_0, 1]] = \gamma_2$, and the canonical representation is such that $B_0 = [0, \bar{b}_0]$, and $B_2 = [\bar{b}_0, 1]$. Because this is a canonical representation of a bi-pooling solution, it must be that $\bar{b}_0 \leq \gamma_1$, as otherwise recommending

[10] B_2 must have nonempty interior, or else incentive compatibility cannot be satisfied.

action a_1 on $[\bar{b}_0, \gamma_1]$ is a profitable deviation. Consequently, this canonical representation is incentive compatible, and hence a commitment outcome is implementable, again a contradiction. Therefore, it must be that B_1 has nonempty interior.

Note also that it must be that $b < \gamma_2$, as otherwise incentive compatibility fails. To see that there exists such an h, I first claim that $\mathbb{E}[\omega \mid \omega \in [0, \gamma_2]] \leq \gamma_1$. Suppose not, so

$$\mathbb{E}[\omega \mid \omega \in [0, \gamma_2]] > \gamma_1. \tag{12}$$

Without loss of generality, assume that there exists a bi-pooling solution features $[y, 1]$ bi-pooled to $\{\gamma_1, \gamma_2\}$ for some $y \in [0, \gamma_1]$.[11] Consequently, there exist $\bar{b}_1$ and $\underline{b}_1$ with $y \leq \underline{b}_1 \leq \bar{b}_1$ such that the (essentially unique) canonical representation of the bi-pooling solution $\{B_0, B_1, B_2\}$ is given by $B_0 = [0, y]$, $B_1 = [\underline{b}_1, \bar{b}_1]$, and $B_2 = [y, \underline{b}_1] \cup [\bar{b}_1, 1]$. Then because $\mathbb{E}[\omega \mid \omega \in B_1] = \gamma_1$ and $\underline{b}_1 \geq 0$, we must have $\bar{b}_1 \leq \gamma_2$ by (12). Thus, the bi-pooling solution must be implementable, a contradiction. As a consequence, it must be that $\mathbb{E}\left[\omega \mid \omega \in [h, b]\right] = \gamma_1$; it remains to show that $h > 0$. If instead $h = 0$, then $\mathbb{E}[\omega \mid \omega \in [0, \gamma_2]] = \gamma_1$. Consequently, it must be that $B_1 = [0, \gamma_2]$ and $B_2 = [\gamma_2, 1]$. This is clearly not optimal: for any $\varnothing \in (0, \gamma_1)$, define $\hat{B}_1 = [\varepsilon, \gamma_2]$, and $\hat{B}_2 = [0, \varepsilon] \cup [\gamma_2, 1]$. Then for ε small enough, $\mathbb{E}[\omega \mid \omega \in \hat{B}_i] \geq \gamma_i$ for each $i = 1, 2$, and Sender's expected payoff is strictly higher. Thus, it must be that $h > 0$.

And to show there exists such an y, it suffices to show that $\mathbb{E}[\omega \mid \omega \in [0, h] \cup [b, 1]] \leq \gamma_2$. Suppose not, so $\mathbb{E}[\omega \mid \omega \in [0, h] \cup [b, 1]] > \gamma_2$. Let $\delta > 0$ be small enough, and let $\epsilon(\delta)$ be such that

$$\mathbb{E}\left[\omega \mid \omega \in [h + \varepsilon(\delta), b - \delta]\right] = \gamma_1.$$

Now define $\tilde{B}_1 = [h + \varepsilon(\delta), b - \delta]$, and $\tilde{B}_2 = [0, h + \varepsilon(\delta)] \cup [b - \delta, 1]]$. Because the density f is strictly positive, for small enough δ, $\mathbb{E}[\omega \mid \omega \in \tilde{B}_2] \geq \gamma_2$, and $\mu_F(\tilde{B}_2) > \mu_F(B_2)$. This creates a profitable deviation to Sender without violating incentive compatibility.

Finally, to show that (11) must hold, suppose to the contrary that

$$v_2 > \frac{\gamma_2 - y}{\gamma_1 - y} v_1.$$

An argument analogous to Case 1 (II) in the proof of Proposition 3.7 shows that Sender has a profitable deviation, and hence the canonical representation $\{B_0, B_1, B_2\}$ such that $B_0 = [0, y]$, $B_1 = [h, b]$, and $B_2 = [y, h] \cup [b, 1]$ cannot be associated to a Sender's preferred

[11] If such y does not exist, by Claim D.3, the bi-pooling solution must be implementable, a contradiction.

ORE. A contradiction. This completes the proof. ∎

By Lemma D.5, Sender's ex ante payoff in her preferred equilibrium can be written as

$$\bar{V}(b) = v(a_1)[F(b) - F(h(b))] + v(a_2)[1 - F(b) + F(h(b)) - F(y(b))],$$

where $h(b)$ and $y(b)$ are implicitly defined by the two equations in (10).

Lemma D.6. *If no commitment outcome is implementable, Sender's ex ante payoff in her preferred equilibrium, $\bar{V}(b)$, is increasing in b.*

Proof of Lemma D.6. To save notation, I write $v_i = v(a_i)$ for $i = 1, 2$. Directly,

$$\bar{V}'(b) = (v_1 - v_2)\left[f(b) - f(h)\frac{\mathrm{d}h}{\mathrm{d}b}\right] - v_2 f(y)\frac{\mathrm{d}y}{\mathrm{d}b}. \tag{13}$$

Using (10), by the implicit function theorem,

$$\frac{\mathrm{d}h}{\mathrm{d}b} = -\frac{(b-\gamma_1)\,f(b)}{(\gamma_1 - h)\,f(h)}, \tag{14}$$

$$\frac{\mathrm{d}y}{\mathrm{d}b} = -\frac{(b-h)\,(\gamma_2-\gamma_1)\,f(b)}{(\gamma_1 - h)\,(\gamma_2 - y)\,f(y)}. \tag{15}$$

Plugging (14) and (15) into (13),

$$\begin{aligned}
\bar{V}'(b) &= (v_1 - v_2)\,f(b)\left(1 + \frac{b-\gamma_1}{\gamma_1 - h}\right) + v_2 f(b)\frac{(b-h)\,(\gamma_2-\gamma_1)}{(\gamma_1-h)\,(\gamma_2-y)} \\
&= \left[(v_1 - v_2)\,\frac{b-h}{\gamma_1-h} + v_2\frac{(\gamma_2-\gamma_1)\,(b-h)}{(\gamma_1-h)\,(\gamma_2-y)}\right] f(b) \\
&= \left[v_1\frac{b-h}{\gamma_1-h} - v_2\frac{(b-h)(\gamma_1-y)}{(\gamma_1-h)(\gamma_2-y)}\right] f(b) \\
&= \left[v_1(\gamma_2-y) - v_2(\gamma_1-y)\right]\frac{(b-h)f(b)}{(\gamma_1-h)(\gamma_2-y)},
\end{aligned}$$

and we see that $\bar{V}'(b) \geq 0$ if and only if $v_1(\gamma_2 - y) \geq v_2(\gamma_1 - y)$. Then since no commitment solution is implementable, by Lemma D.5, (11) implies that $\bar{V}'(b) \geq 0$, and hence Sender's ex ante payoff in her preferred equilibrium is increasing in b. ∎

E Equilibrium Refinement

In this section, I show that any obedient recommendation equilibrium (ORE) defined in Section 4 survives the Never-a-Weak-Best-Response (NWBR) criterion proposed by Cho and Kreps (1987). I use the term "type" instead of "state" henceforth to make exposition easier.

I introduce some notation first. For any $m \in \mathcal{C}$, let $MBR(m)$ denote the set of all mixed strategy best responses for Receiver to message m for any belief $p(\cdot \mid m)$.[12] And for a fixed ORE, let v^*_θ denote the equilibrium payoff of type θ. Finally, for a fixed ORE and $m \notin \{B_0, \dots, B_{n-1}\}$, where $\{B_0, \dots, B_{n-1}\}$ is the deterministic representation corresponding to the ORE, define

$$D(\theta, m) = \left\{ \rho \in MBR(m) \,:\, v^*_t < \sum_a v(a)\rho(a) \right\},$$

and

$$D^0(\theta, m) = \left\{ \rho \in MBR(m) \,:\, v^*_t = \sum_a v(a)\rho(a) \right\};$$

in words, $D(\theta, m)$ is the set of mixed strategy best responses that make type θ strictly prefer m to her equilibrium message, and $D^0(\theta, m)$ is the set of mixed strategy best responses that make type θ exactly indifferent.

For every $m \notin \{B_0, \dots, B_{n-1}\}$, let $\ell = \min\{i \,:\, m \cap B_i \neq \emptyset\}$. By changing the belief restriction to

$$p\left(\min m \cap B_\ell \mid m\right) = 1, \tag{16}$$

I can prove that any ORE survives the NWBR criterion. By the definition of ORE, Receiver never mixes on path, and hence for any $\theta' \in [0,1]$, $D^0(\theta', m) = \{a_k\}$ if and only if $\theta \in B_k \setminus (\cup_{i>k} B_i)$. Furthermore, define

$$M = \{a_k \,:\, k \text{ is such that } m \cap B_k \neq \emptyset\};$$

then the lowest action in M is a_ℓ. Now for any $\theta' \in [0,1]$,

$$\bigcup_{\theta \neq \theta'} D(\theta, m) = \{\rho \in \Delta(M) \,:\, \operatorname{supp} \rho \subseteq \{a_k, a_{k+1}\} \text{ with } k \geq \ell, \text{ and } \rho(a_k) < 1\}.$$

Then $D^0(\theta', m) \subseteq \cup_{\theta \neq \theta'} D(\theta, m)$ if and only if $D^0(\theta', m) = \{a_k\}$ with $k > \ell$, which is in

[12] Note that p must satisfy $\operatorname{supp} p(\cdot \mid m) \subseteq m$.

turn equivalent to $\theta' \in \cup_{k>\ell}(B_k \setminus (\cup_{i>k} B_i))$. Then (16) implies that $\theta' \notin \operatorname{supp} p(\cdot \mid m)$. Consequently, any ORE must survive the NWBR criterion.

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