Withholding Verifiable Information

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Abstract

I study a class of verifiable disclosure games where the sender's payoff is state independent and the receiver's optimal action only depends on the expected state. In such games, what is the sender's preferred equilibrium? When does the sender gain nothing from having commitment power? I identify conditions for an information design outcome to be an equilibrium outcome of the verifiable disclosure game, and give simple sufficient conditions under which the sender does not benefit from commitment power. These results help me to characterize the sender's preferred equilibria and her equilibrium payoff set in a class of verifiable disclosure games. I apply these insights to study influencing voters and selling with quality disclosure.

Keywords: Communication, Verifiable disclosure games, Sender's preferred equilibrium, Information design
JEL Classifications: C72; D82; D83

1 Introduction

In many economic interactions that involve communication between a sender and a receiver, the sender's messages are verifiable in the sense that they can be vague but can never be false. Examples include sellers disclosing and highlighting certain features of a

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product to consumers, political experts organizing and simplifying poll results for politicians, and advisors condensing and distilling market research for their managers. In each of these examples, if the sender releases information that is inherently false, she would face severe consequences: a seller might be sued for compensation, and an expert or advisor could be fired for lying.

I study a communication game in which the sender's messages are verifiable, a so-called *verifiable disclosure game*, and emphasize the extent to which the sender benefits from communication. It is well known that the unraveling equilibria, in which the sender's private information is fully revealed, are the receiver's preferred equilibria (Grossman, 1981; Milgrom, 1981). However, little is known about the sender's preferred equilibria. How much can the sender benefit from verifiable communication? What are the structures of the sender's preferred equilibria?

In this paper, I find conditions under which the sender's payoff in her preferred equilibrium is the same as if she could commit to an arbitrary information structure. Then, I develop tools to characterize the sender's preferred equilibrium if these conditions are not satisfied. I also discuss other equilibria of the game as well as the sender's equilibrium payoff set. I apply my findings to two specific settings: I study how a seller should optimally reveal product information and how an expert should communicate with voters.

In the model, there is a one-dimensional state of the world that is payoff-relevant to the receiver and only observable to the sender. The sender sends a message, and the receiver subsequently chooses an action from a finite set. The actions are "ordered" in the sense that the receiver would like to use a higher action when the expected state is higher, and the sender strictly prefers a higher action regardless of the state. Verifiability requires that the sender's messages must contain the true state. Finally, the sender's payoff depends only on the receiver's action (in jargon, the sender's preferences are state-independent), and the receiver's optimal choice of action depends only on the expected state.

A natural upper bound of the sender's expected payoff in the sender's preferred equilibrium is given by the solution to the corresponding information design problem. The information design approach, popularized by Kamenica and Gentzkow (2011), assumes that the sender has commitment power: she can commit to any information structure that reveals information about the state. Kleiner, Moldovanu, and Strack (2021) and Arieli, Babichenko, Smorodinsky, and Yamashita (2022) show that every information design problem admits a salient solution, a so-called *bi-pooling solution*, that has a simple structure and is easy to identify. If a bi-pooling solution can be induced by an equilibrium, it must be a sender's preferred equilibrium, and the sender's payoff in this equilibrium coincides with the commitment payoff.

My first set of results identifies necessary and sufficient conditions under which a bi-pooling solution is an equilibrium outcome of the verifiable disclosure game. These results suggest a "guess and verify" approach for finding the sender's preferred equilibrium: one can solve the corresponding information design problem first, and then check whether the conditions hold. I first note that every bi-pooling solution can be represented by a partition of the state space such that the same action is recommended in all states belonging to the same partitional element. Furthermore, each partitional element is either an interval or obtained from "subtracting" an interval from a larger interval. Among other things, I find that a bi-pooling solution is an equilibrium outcome if and only if in each state the sender prefers to use the partitional element that contains the state as her message rather than fully disclose the state.

My second set of results provides sufficient conditions on model primitives under which commitment has no value; i.e., that imply the commitment payoff can be achieved in an equilibrium. The first condition requires that it is sufficiently more profitable for the sender to induce the higher action than the lower one for any pair of adjacent actions, which guarantees that it is never optimal for the sender to recommend a lower action in a state in which a higher action is optimal. Another sufficient condition, which is implied by the first one, identifies separate conditions on the prior distribution and the sender's value function. Importantly, under any of these conditions, the sender's preferred equilibrium of a verifiable disclosure game can be obtained by solving the corresponding information design problem. Moreover, for any information design problem that satisfies one of these conditions, the commitment assumption can be dispensed with.

Even if the commitment payoff cannot be attained in any equilibrium, I show that the sender's preferred equilibrium exists, and I characterize its properties. In particular, in this equilibrium, every on-path message of the sender can be interpreted as an action recommendation that the receiver finds optimal to follow, and the messages have a simple structure. Furthermore, the sender's expected payoff in this equilibrium is strictly higher than the unraveling payoff.[1] Moreover, I find that any payoff that is below the sender's payoff in her preferred equilibrium and above the unraveling payoff can be sustained in an equilibrium in which the on-path messages have the properties described above.

[1] "Unraveling payoff" refers to the sender's payoff in any equilibrium that features unraveling: in my model, all such equilibria generate the same expected payoff to the sender.

I apply these insights to study selling with information disclosure and influencing voters. In the selling setting, Milgrom (1981) shows that when the buyer can buy any fraction of the product, every equilibrium of the game features unraveling. Perhaps surprisingly, if the buyer is restricted to buying integer units, the seller may be able to achieve the same outcome as if she has commitment power. Next, I consider an expert who discloses verifiable information to a group of voters in an amendment voting setting; that is, voters choose from one of the three alternatives: the amended bill, the (unamended) bill, and no bill (or status quo). Interestingly, the expert can be hurt even if, all else equal, all voters are more inclined toward the expert's most preferred outcome.

Finally, I consider two extensions of the baseline model: in one of them I allow the state to be multidimensional, and in another I drop the "orderedness" of actions and further allow the receiver's action space to be an interval. For both extensions, I find necessary and sufficient conditions under which a bi-pooling solution (or, when the state is multidimensional, its natural generalization) is an equilibrium outcome. I also identify economically meaningful communication environments in which either commitment has no value, or the commitment payoff cannot be obtained in any equilibrium.

Related literature. As mentioned above, the study of verifiable disclosure games initiated from Grossman and Hart (1980), Grossman (1981), and Milgrom (1981); for surveys of this literature, see Milgrom (2008) and Dranove and Jin (2010). As far as I know, there are very few papers studying the sender's preferred equilibria in verifiable disclosure games. Ali, Lewis, and Vasserman (2022) consider a model of personalized pricing with disclosure of verifiable payoff-relevant information by a consumer, and they study the highest consumer surplus that can be achieved in equilibrium under different assumptions on the consumer's message space. The work closest to mine is Titova (2022): she shows that the sender can attain the information design payoff in an equilibrium of the verifiable disclosure game when there are two actions. Titova also assumes that the sender has state-independent preferences, and her assumptions on state space and message space are essentially the same as mine, though she allows that the receiver's actions are "unordered".[2]

[2] In a more recent version of Titova (2022), the author also looks at the case that the receiver has more than two (but finitely many) actions. Titova proves a result related to my Proposition 3.4: in words, if an information design solution admits a deterministic representation that is incentive compatible, then this information design solution is implementable. By focusing on bi-pooling solutions, I am able to show that (1) every bi-pooling solution admits a unique canonical representation, and (2) a bi-pooling solution is implementable *if and only if* its canonical representation is incentive compatible.

This work is also connected to the literature on the sender's preferred equilibria in cheap talk games. Assuming that the sender has state-independent preferences, Lipnowski and Ravid (2020) characterizes the set of equilibrium payoffs in a cheap talk game, and they show that the sender's preferred equilibrium payoff is given by the quasiconcave hull of the sender's value function evaluated at the prior. Lipnowski (2020) shows that in a cheap talk game with finite parameters, a sufficient condition for the sender to attain the same payoff as under communication with commitment is that her value function is continuous.

Finally, this paper is related to the literature on relaxing the commitment assumption in information design problems. Lipnowski, Ravid, and Shishkin (2021) and Min (2021) allow the sender to stochastically and costlessly alter her message after observing the state. In Nguyen and Tan (2021), the sender can costly distort the messages produced by the information structure she chooses. Lin and Liu (2022) consider a sender who can secretly deviate to some other information structures so long as the distribution over messages is the same as the announced one. My model differs in that the only constraint that the sender faces is that her messages have to be verifiable.

2 The Model

There are two players, Sender and Receiver. The state space is $\Omega = [0, 1]$, and Sender and Receiver share a common prior F, which admits a strictly positive density f. Let μ_F denote the probability measure induced by F. I assume that Receiver's optimal action only depends on the posterior mean of the state, denoted by x. Receiver has $n < \infty$ actions, and hence her action space is $\mathcal{A} = \{a_0, a_1, \dots, a_{n-1}\}$. There exist cutoffs $0 = \gamma_0 \leq \gamma_1 \leq \cdots \leq \gamma_n = 1$ such that a_i is optimal if and only if $x \in A_i := [\gamma_i, \gamma_{i+1}]$. Sender's payoff $v : \mathcal{A} \to \mathbb{R}$ only depends on Receiver's action; normalize $v(a_0) = 0$, and assume $v(a_k) > v(a_l)$ whenever $k > l$.

Sender observes the state and sends a message to Receiver. Assume that for every $\omega \in [0, 1]$, Sender's message space is $\mathcal{M}(\omega) = \{m \in \mathcal{C} : \omega \in m\}$, where $\mathcal{C}$ denotes the collection of nonempty closed subsets of $[0, 1]$. In this setting Sender and Receiver play a **verifiable disclosure game**, first studied by Grossman and Hart (1980), Grossman (1981), and Milgrom (1981); Sender's message space in my problem is also identical to the aforementioned papers. Any message is verifiable in the sense that it contains the true state. For each state $\omega \in [0, 1]$, the assumption on message space allows Sender to fully reveal the state, namely sending $m = \{\omega\}$, or to disclose nothing, that is sending $m = [0,1]$.

Formally, the verifiable disclosure game is defined as follows. Sender's strategy set, Σ, is the set of functions $\sigma : [0,1] \to \Delta(\mathcal{C})$ such that $\sigma(\omega)$ is supported on $\mathcal{M}(\omega)$ for each $\omega \in [0,1]$.[3] That is, if Sender uses message m with positive probability in state ω, then it must be that m is feasible in this state. Receiver's strategy set, T, is the set of functions $\tau : \mathcal{C} \to \Delta(\mathcal{A})$. A belief system for Receiver, $p : \mathcal{C} \to \Delta([0,1])$, yielding Receiver's beliefs about the state as a function of the observed message.

Say that (σ, τ, p) is a (perfect Bayesian) **equilibrium** if the following conditions hold. First, $\sigma(\omega)$ is supported on $\arg\max_{m' \in \mathcal{M}(\omega)} v(\tau(m'))$. Second, for every $m \in \mathcal{C}$ and $a_i \in \mathcal{A}$, $\tau(a_i \mid m) > 0$ implies $\mathbb{E}[\omega \mid \omega \in m] \in [\gamma_i, \gamma_{i+1}]$. Third, for every $\omega \in [0,1]$ and $m \in \mathcal{C}$, $\omega \notin \operatorname{supp}(p(m))$ if $m \notin \mathcal{M}(\omega)$.[4] That is, Receiver must deem any state in which the observed message m cannot be sent impossible. Finally, p is obtained from the prior F, given σ, using Bayes' rule. An **outcome** of this verifiable disclosure game is a pair $(G, R) \in \Delta([0,1]) \times \mathbb{R}$, where G is the induced distribution over Receiver's posterior means, and R is Sender's ex ante payoff. An outcome is an **equilibrium outcome** if it is induced by an equilibrium.

A bulk of this paper concerns Sender's preferred equilibria of this verifiable disclosure game, which are the equilibria that attain the highest possible Sender's ex ante payoff. For this purpose, it is useful to introduce Sender's value function $u : [0,1] \to \mathbb{R}$, which maps every posterior mean x to the highest attainable expected Sender payoff, conditional on the posterior mean being x and Receiver choosing her optimal action. Figure 1 illustrates Sender's value function when there are four actions. By definition, $u(x)$ is constant on (γ_i, γ_{i+1}), and has discontinuous at γ_i for each $i = 1, 2, \ldots, n-1$. Moreover, it is upper semicontinuous.

To study Sender's preferred equilibria and the value of commitment, I also consider the information design problem based on this communication environment as a benchmark. In this problem, Sender can commit to any experiment that reveals information about the state. An **experiment** (χ, S) consists of a **signal space** S and a mapping $\chi : [0,1] \to \Delta(S)$; an experiment induces a **signal distribution** in $\Delta(S)$. For each state $\omega \in [0,1]$, $\chi(\omega)$ identifies a distribution over signal space S; an experiment is **deterministic** if $\chi(\omega)$ is a degenerate distribution for each $\omega \in [0,1]$. Because Receiver's optimal action only depends on the posterior mean of the state, it is without loss to restrict attention on the

[3]For a compact metric space Y, let $\Delta(Y)$ denote the set of all probability measures on the Borel subsets of Y. Endowing $\mathcal{C}$ with the Hausdorff distance, it is a compact metric space.

[4]The support of a distribution, denoted by $\operatorname{supp}(\cdot)$, is the smallest closed set that has probability one.

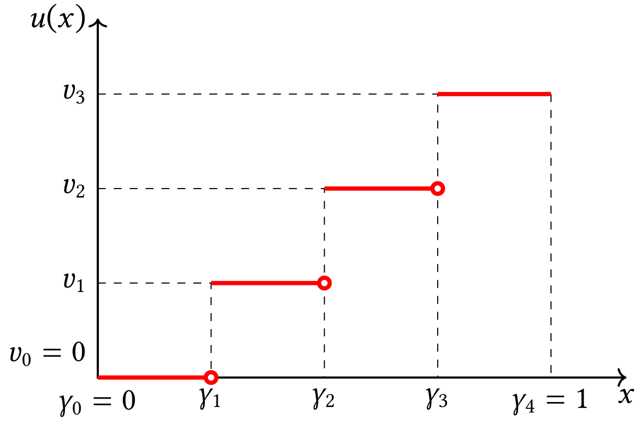

Figure 1: An illustration for Sender's value function, which is a function of posterior mean x, when there are four actions.

class of experiments where $S = [0, 1]$, and each $s \in S$ equals the induced posterior mean: $s = \mathbb{E}[\omega \mid s]$. It is well known that[5] a signal distribution G is induced by an experiment if and only if G is a mean-preserving contraction of F;[6] consequently, Sender's problem is to choose a posterior mean distribution that solves

$$\max_{G \in \mathrm{MPC}(F)} \int_0^1 u(x)\, \mathrm{d}G(x), \tag{1}$$

where $\mathrm{MPC}(F)$ is the set of mean-preserving contractions of F. I call any solution G_c to problem (1) an **optimal signal distribution**, and call the value of problem (1), denoted by R_c, the **commitment payoff**. Clearly, R_c is an upper bound of Sender's equilibrium payoff in the verifiable disclosure game.

Say that a pair $(G, R) \in \Delta([0, 1]) \times \mathbb{R}$ is a **commitment outcome** if G is an optimal signal distribution and R is Sender's commitment payoff. A commitment outcome (G, R) is **implementable** by verifiable messages, or just implementable for short, if (G, R) is an equilibrium outcome of the verifiable disclosure game based on the same communication environment. I usually abuse notation and say that an optimal signal distribution G is implementable when the commitment outcome it induces is implementable.

[5] See, for example, Gentzkow and Kamenica (2016) and Kolotilin (2018).

[6] A distribution $G \in \Delta([0, 1])$ is a mean-preserving contraction of F if $\int_0^x G(s)\mathrm{d}s \leq \int_0^x F(s)\mathrm{d}s$ for all $x \in [0, 1]$, where the inequality holds with equality at $x = 1$.

3 Implementability

There are two sets of results presented in this section. The first set of results (Section 3.3) provides necessary and sufficient conditions under which a particular commitment outcome can be implemented. The second set of results (Section 3.4) provides sufficient conditions on model primitives under which commitment has no value. To get there, I first characterize a salient class of solutions to the information design problem; this is handled in Section 3.1 and Section 3.2.

3.1 Characterizing Information Design Solution

I start by introducing a class of simple signal distributions.

Definition 1 (Bi-pooling distribution)**.** A distribution $G \in \mathrm{MPC}(F)$ is a **bi-pooling distribution** if there exists a collection of pairwise disjoint intervals $\left\{ \left[\underline{\omega}_i, \overline{\omega}_i\right] \right\}_{i \in \mathcal{I}}$ such that

- for all $i \in \mathcal{I}$, $G(\overline{\omega}_i) - G\left(\underline{\omega}_i\right) = F(\overline{\omega}_i) - F\left(\underline{\omega}_i\right)$ and $\left|\mathrm{supp}\left(G|_{[\underline{\omega}_i, \overline{\omega}_i]}\right)\right| \le 2$;[7]
- $G|_{[0,1] \setminus \cup_{i \in \mathcal{I}} \left[\underline{\omega}_i, \overline{\omega}_i\right]} = F|_{[0,1] \setminus \cup_{i \in \mathcal{I}} \left[\underline{\omega}_i, \overline{\omega}_i\right]}$.

Intuitively, a bi-pooling distribution partitions the state space in intervals such that in each of them,

(i) either all states are fully revealed,
(ii) or a single deterministic signal is sent for all states,
(iii) or two different, possibly random, signals are sent.

In particular, $\left[\underline{\omega}_i, \overline{\omega}_i\right]$ is called a **pooling interval** if $\left|\mathrm{supp}\left(G|_{[\underline{\omega}_i, \overline{\omega}_i]}\right)\right| = 1$ (which corresponds to case (ii) above), and it is called a **bi-pooling interval** if $\left|\mathrm{supp}\left(G|_{[\underline{\omega}_i, \overline{\omega}_i]}\right)\right| = 2$ (which corresponds to case (iii) above).

Kleiner et al. (2021) and Arieli et al. (2022) show that every extreme point of MPC(F) is a bi-pooling distribution. Then because MPC(F) is convex and compact, and the objective function in problem (1) is linear and upper semicontinuous, by the Bauer's Maximum Principle,[8] an extreme point of MPC(F) solves problem (1). Calling a bi-pooling distribution G_B that solves the information design problem (1) a **bi-pooling solution**,

Theorem 3.1 (Kleiner et al. 2021; Arieli et al. 2022)**.** *The information design problem admits a bi-pooling solution.*

[7] For any $H \in \mathrm{MPC}(F)$, let $H|_{[c,d]}$ denote the restriction of G to $[c, d] \subseteq [0, 1]$. Moreover, for a finite set K, let $|K|$ denote the cardinality of K.

[8] See, for example, Theorem 7.69 in Aliprantis and Border (2006).

Applying Theorem 3.1 to my setting, Lemma 3.2 characterizes the optimal signal distribution of Sender's problem when she has commitment power.

Lemma 3.2 (Candogan, 2019)**.** *Every bi-pooling solution to the information design problem satisfies* $\text{supp}\,(G_B) \subseteq [0, \tilde{\gamma}] \cup \{\gamma_i\}_{i=1}^{n-1}$*, where* $\tilde{\gamma} \in [0, 1]$ *is such that* $\tilde{\gamma} \leq \gamma$ *for all* $\gamma \in \text{supp}\,(G_B) \cap \{\gamma_i\}_{i=1}^{n-1}$*.*

An important consequence of Lemma 3.2 is that every signal realization can be interpreted as a recommended action. In particular, every $x \in \text{supp}\,(G_B) \cap [0, \tilde{\gamma}]$ recommends the "lowest" action,[9] and γ_i is a recommendation of action a_i for each $i = 1, 2, \ldots, n-1$. Another appealing property of bi-pooling solutions is that they can be solved via a finite-dimensional convex program proposed by Candogan (2019), which admits a polynomial algorithm.[10]

3.2 Deterministic Representations of a Bi-Pooling Solution

It is tempting to ask whether every bi-pooling solution can be induced by a deterministic experiment. To answer this, I introduce some notation first. Say that a bi-pooling interval $[\underline{\omega}, \overline{\omega}]$ of a bi-pooling solution G_B with $\text{supp}\,\left(G_B|_{[\underline{\omega}, \overline{\omega}]}\right) = \{z_L, z_H\}$ where $z_L \leq z_H$ admits a **bi-partition** $\{B_L, B_H\}$ if there exist two closed sets B_L and B_H,[11] such that

- $B_L \cup B_H = [\underline{\omega}, \overline{\omega}]$;
- $\mu_F\,(B_L \cap B_H) = 0$;
- $\mathbb{E}[\omega \mid \omega \in B_L] = z_L$, and $\mathbb{E}[\omega \mid \omega \in B_H] = z_H$.

If a bi-pooling interval of a bi-pooling solution admits a bi-partition $\{B_L, B_H\}$, then for $i = H, L$ and every $\omega \in \text{int}\,(B_i)$,[12] $\text{supp}\,(\chi(\omega)) = \{z_i\}$. In words, in every state that belongs to a component of the bi-partition, a deterministic signal, which is the conditional mean of the state on that component, realizes with probability one.

If each of the bi-pooling intervals admits a bi-partition, by Lemma 3.2, a bi-pooling solution G_B can be represented by a sequence of closed sets $\{B_i\}_{i=0}^{n-1}$ such that[13]

(I) $\cup_{i=0}^{n-1} B_i = [0, 1]$;

[9] For example, if a_0 is never recommended, the "lowest" action is a_1.

[10] That is, the number of computational steps in the algorithm can be expressed as a polynomial of n, the number of actions.

[11] For a function $g : [0, 1] \to \mathbb{R}$ and $Y \subseteq [0, 1]$, let $g|_Y$ denote the restriction of g to Y.

[12] For a subset S of $[0, 1]$, $\text{int}\,(S)$ is the interior of S.

[13] I allow some of the elements of this sequence to be empty sets.

(II) $\mu_F\left(B_i \cap B_j\right) = 0$ for any $i, j = 0, \dots, n-1$; and

(III) if let $j = \min\{i : \operatorname{int}(B_i) \neq \emptyset\}$, then $\mathbb{E}[\omega \mid \omega \in B_j] \geq \gamma_j$ and $\mathbb{E}[\omega \mid \omega \in B_i] = \gamma_i$ for all $i > j$.

Then for every $i = 0, \dots, n-1$, $B_i \backslash (\cup_{j>i} B_j)$ is the set of states in which action a_i is recommended with probability one; and by (III), every recommendation is obedient in the sense that the receiver's optimal action coincides with the action recommendation.[14] Say that this bi-pooling solution admits a **deterministic representation** $\{B_i\}_{i=0}^{n-1}$.

Lemma 3.3, which is largely based on Lemma 4 in Arieli et al. (2022), shows that every bi-pooling solution has a particular deterministic representation. Say that a bi-partition $\{B_L, B_H\}$ of a bi-pooling interval $[\underline{\omega}, \overline{\omega}]$ is a **nested intervals representation** if there exist $\underline{b}_L, \overline{b}_L \in [\underline{\omega}, \overline{\omega}]$ such that $B_L = [\underline{b}_L, \overline{b}_L]$ and $B_H = [\underline{\omega}, \underline{b}_L] \cup [\overline{b}_L, \overline{\omega}]$.

Lemma 3.3. *Let $[\underline{\omega}, \overline{\omega}]$ be a bi-pooling interval of a bi-pooling solution G with* $\operatorname{supp}\left(G|_{[\underline{\omega}, \overline{\omega}]}\right) = \{\gamma_L, \gamma_H\}$ *where $\gamma_L < \gamma_H$, there exists a unique nested interval representation $\{B_L, B_H\}$ of $[\underline{\omega}, \overline{\omega}]$.*

Unless otherwise noted, the proofs are collected in Appendix A. With the help of Lemma 3.3, a bi-pooling solution can be represented by a deterministic representation $\{B_i\}_{i=0}^{n-1}$, where for $i = 1, \dots, n-1$, if $\operatorname{int}(B_i) \neq \emptyset$, B_i is either an interval or the union of two intervals.[15] I call such a deterministic representation of a bi-pooling solution a **canonical representation**; uniqueness of the nested interval representation of bi-pooling intervals begets the essential uniqueness of the canonical representation.[16]

3.3 Characterizing Implementability

The first set of main results characterizes the implementability of any bi-pooling solution.

Definition 2. A deterministic representation $\{B_i\}_{i=0}^{n-1}$ is **incentive compatible** if deviating to full disclosure is never profitable:

$$A_i \subseteq \bigcup_{j=i}^{n-1} B_j \tag{IC}$$

[14] To be sure, it cannot be that $\mathbb{E}[\omega \mid \omega \in B_j] \geq \gamma_{j+1}$, as otherwise $\sup B_j > \gamma_{j+1}$ and it is strictly more profitable for Sender to recommend a_{j+1} instead on $[\gamma_{j+1}, \sup B_j]$.

[15] This can be obtained from "subtracting" an interval from a larger interval.

[16] By "essential uniqueness" I mean that if $\{B_i\}_{i=0}^{n-1}$ and $\{\hat{B}_i\}_{i=0}^{n-1}$ are two canonical representations of a bi-pooling solution, $B_j = \hat{B}_j$ for all j such that $\mu_F(B_j) > 0$. In words, they can only differ on elements that are null sets.

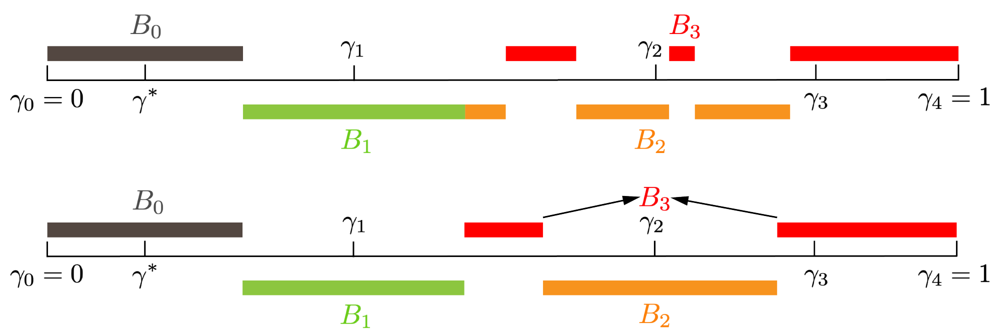

Figure 2: A deterministic representation (upper panel) and a canonical representation (lower panel) of a bi-pooling solution when there are four actions. In the canonical representation displayed in the lower panel, $\{B_2, B_3\}$ is a nested interval representation.

for all $i = 0, 1, \dots, n-1$.

To understand (IC), consider any state $\omega \in A_i$. In this state, Receiver would take action a_i under complete information, that is, when the state is fully revealed. Since $A_i \subseteq \cup_{j \geq i} B_j$, in state ω the deterministic representation recommends an action at least as preferred as a_i with probability one, and Receiver obeys this recommendation. Consequently, deviating to full disclosure is never profitable.

Proposition 3.4. *A bi-pooling solution is implementable if and only if its canonical representation $\{B_i\}_{i=0}^{n-1}$ is incentive compatible.*

Proposition 3.4 says that to check whether a bi-pooling solution is implementable, it suffices to find its (essentially unique) canonical representation, and see if Condition (IC) is satisfied. This result is useful because one only needs to focus on a specific messaging strategy of Sender, and only one kind of deviation needs to be considered, namely deviating to full disclosure.

Proposition 3.4 is proved in three steps. In the first step, I show that every implementable bi-pooling solution can be induced by an equilibrium in which both players only use pure strategies. To see this, note that because Receiver only mixes between two adjacent actions, if in a state Sender uses more than one message with positive probability, it must be that they induce the same distribution over Receiver's actions upon receiving the messages. Therefore, in any state, one can "merge" the multiple messages used with positive probability into one by taking the union. On Receiver's side, recall that the commitment payoff is an upper bound of the equilibrium payoff set. Hence, in any equilibrium whose outcome coincides with the commitment outcome Receiver must always break ties in favor of Sender whenever she is indifferent. Consequently, in this equilibrium, every state maps to a single action with probability 1, and hence it can be identified by a deterministic representation.

In the second step, I show that a bi-pooling solution is implementable if and only if there exists a deterministic representation of it that is incentive compatible. The "only if" direction is an immediate consequence of the first step: if a bi-pooling solution is implementable, its outcome is achieved in an equilibrium that can be identified by a deterministic representation. Then it must be that Sender does not want to deviate to full disclosure in any state, and thus the associated deterministic representation must satisfy Condition (IC). For the other direction, it suffices to show that for the incentive compatible deterministic representation $\{B_i\}_{i=0}^{n-1}$, Sender's strategy σ given by $\sigma(B_i \,|\, \omega) = 1$ for all $\omega \in B_i \backslash (\cup_{j>i} B_j)$ can be sustained in an equilibrium. To this end, consider the *maximally skeptical beliefs*: for any off-path message $m \notin \{B_i\}_{i=0}^{n-1}$, $p(\min m \,|\, m) = 1$. In words, Receiver believes that the state is the lowest possible one when she sees an off-path message. Then because each message must contain the true state, any off-path message is no better than full disclosure. Therefore, Condition (IC) is also sufficient.

Finally, I show that the canonical representation is the most "deviation proof" one amongst all deterministic representations: if there exists an implementable deterministic representation, the (essentially unique) canonical representation of the same bi-pooling solution must be implementable. Intuitively, suppose $\{\gamma_L, \gamma_H\}$ is bi-pooled to $[\underline{\omega}, \overline{\omega}]$; in the canonical representation, B_L is the most "concentrated around" γ_L amongst all deterministic representations, and hence minimizes the chance of B_L intersecting A_j's with $j \geq L$.

The following result goes one step further: it reveals the exact features of the canonical representation that make Condition (IC) fail and hence prevent the bi-pooling solution from being implementable. Say that an action a_i with $i \geq 1$ is **skipped** in a deterministic representation $\{B_i\}_{i=0}^{n-1}$ if $\text{int}\,(B_i) = \emptyset$. That is, from an ex ante perspective, this action is recommended with probability zero. For each action i that is not skipped, let $\bar{b}_i = \sup B_i$, and $\underline{b}_i = \inf B_i$.

Proposition 3.5. *A bi-pooling solution is implementable if and only if its canonical representation is such that*

(i) for $i \geq 1$, if a_i is skipped, $\overline{b}_j \leq \gamma_i$ for every unskipped action a_j with $j < i$, and

(ii) for every nested intervals representation $\{B_L, B_H\}$, $\overline{b}_L \leq \gamma_H$.

For the canonical representation of an arbitrary bi-pooling solution, Condition (i) says

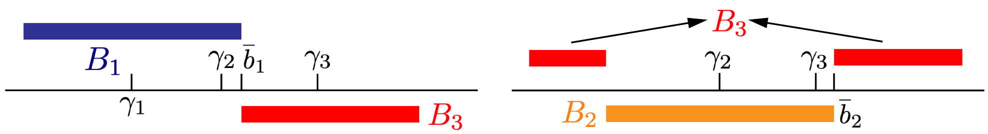

Figure 3: Violation of the conditions in Proposition 3.5: in the left panel condition (i) is violated because a_2 is skipped, but $\gamma_2 < \bar{b}_1$ where a_1 is a "lower" action that is not skipped; in the right panel condition (ii) is violated since $\{B_2, B_3\}$ is a nested intervals representation, but $\bar{b}_2 > \gamma_3$.

that if action a_i is skipped, it must be that no "lower" action is recommended in any state in which a_i is optimal under complete information. Condition (ii) states that if two unskipped actions, a_L and a_H with $L < H$, are such that $\{B_L, B_H\}$ is a nested interval representation, then the highest state in which the lower action a_L is recommended lies "below" the lowest state in which the higher action a_H is optimal under complete information. It is easy to see that Conditions (i) and (ii) are necessary: if any of them is violated, Sender prefers to deviate to full disclosure in a set of states of positive measure. The special structure of the canonical representation of a bi-pooling solution guarantees that the two conditions are also sufficient for Sender not to deviate to full disclosure in any state. Therefore, the two conditions are equivalent to Condition (IC).

An important implication of Proposition 3.5 is that deviating to full disclosure is profitable only when an action a_i with $i < n - 1$ is induced "too frequently" in the canonical representation; this is illustrated in Figure 3. Furthermore, this "threat" is only relevant for skipped actions and nested intervals. The two conditions in Proposition 3.5 are easier to check than Condition (IC) in determining whether a bi-pooling solution is implementable, and they allow me to identify sufficient conditions under which commitment has no value in the next subsection.

Corollary 3.6, which is a direct consequence of Proposition 3.5, concerns the special case that there are two actions.

Corollary 3.6. *If $|\mathcal{A}| = 2$,[17] then every bi-pooling solution can be implemented.*

Corollary 3.6 is very similar to Theorem 2 in Titova (2022). Intuitively, in an information design problem with two actions, Sender's lone objective is to maximize the probability that the "high action" is played, and hence she only needs to pool as many low states (states in A_0) with high states (states in A_1) as possible. Therefore, for all type $\omega \in A_1$, it must be that $\omega \in B_1$. Consequently, incentive compatibility holds for the (essentially

[17] For a finite set F, denote its cardinality by $|F|$.

unique) canonical representation of every bi-pooling solution, and hence they are implementable.

In the following example, taken from Gentzkow and Kamenica (2016), the two conditions in Proposition 3.5 are satisfied, and the commitment outcome is hence implementable.

Example 1 (Gentzkow and Kamenica, 2016)**.** Suppose $\mathcal{A} = \{a_0, a_1, a_2\}$, and $\gamma_1 = 1/3$, $\gamma_2 = 2/3$; that is, $A_0 = [0, 1/3]$, $A_1 = [1/3, 2/3]$, and $A_2 = [2/3, 1]$. The prior F is uniform. Assume also that $v(a_1) = 1$ and $v(a_2) = 3$. Gentzkow and Kamenica (2016) show that

$$B_0 = [0, 8/48]\,,\ B_1 = [11/48, 21/48]\,,\ B_2 = [8/48, 11/48] \cup [21/48, 1]$$

is the canonical representation of a bi-pooling solution to the information design problem.[18] Because no action is skipped, and the nested intervals representation $\{B_1, B_2\}$ satisfies $\overline{b}_1 \leq \gamma_2$, by Proposition 3.5, this bi-pooling solution can be implemented. Consequently, Sender's commitment payoff can be attained in an equilibrium of the verifiable disclosure game.

Example 2 and Example 3 illustrate that for any bi-pooling solution, if one of the conditions in Proposition 3.5 fails to hold, it cannot be implemented. In particular, these examples are obtained by "slightly perturbing" the binary actions environment.

Example 2. Suppose $\mathcal{A} = \{a_0, a_1, a_2\}$, and $\gamma_1 = 0.5$, $\gamma_2 = 0.9$; that is, $A_0 = [0, 0.5]$, $A_1 = [0.5, 0.9]$, and $A_2 = [0.9, 1]$. The prior F is uniform. Assume that $v(a_1) = 1$ and $v(a_2) = 1.1$. The unique solution to the information design problem has an essentially unique canonical representation $\{B_0, B_1, B_2\}$ with $B_1 = [0, 1]$ and $B_0 = B_2 = \emptyset$. Action a_2 is skipped, and $0.9 = \gamma_2 < \bar{b}_1 = 1$, which violates Condition (i) in Proposition 3.5. Therefore, this bi-pooling solution cannot be implemented.

When there are three actions, as pointed out by Gentzkow and Kamenica (2016), in the information design problem Sender needs to trade-off between how frequently a_1 and a_2 are taken: inducing a_2 more frequently decreases the frequency that a_1 is taken. In Example 2, the gap between $v(a_1)$ and $v(a_2)$ is much smaller than the gap between $v(a_0)$ and $v(a_1)$, hence it is more profitable to induce more a_1 than a_2. Consequently, the unique information design solution recommends a_1 with probability one. Then for $\omega > \gamma_2$, Sender

[18] Lemma C.4 in the Supplementary Appendix shows that all bi-pooling solutions have the same canonical representation ("essentially" means that two canonical representations can only differ B_i's that are μ_F-null).

is strictly better off by fully revealing the state, hence the information design solution is not implementable.

Example 3. Suppose $\mathcal{A} = \{a_0, a_1, a_2\}$, and $\gamma_1 = 0.6$, $\gamma_2 = 0.7$; that is, $A_0 = [0, 0.6]$, $A_1 = [0.6, 0.7]$, and $A_2 = [0.7, 1]$. The prior F is uniform. Assume $v(a_1) = 1$ and $v(a_2) = 1.3$. Solving the information design problem, it can be seen that

$$B_0 = [0, 0.267],\ B_1 = [0.356, 0.845],\ \text{and } B_2 = [0.267, 0.356] \cup [0.845, 1]$$

is the canonical representation of a bi-pooling solution. No action is skipped, and $\{B_1, B_2\}$ is a nested interval representation. Since $\bar{b}_1 = 0.845 > 0.7 = \gamma_2$, Condition (ii) in Proposition 3.5 fails to hold, and thus this bi-pooling solution cannot be implemented.

3.4 Sufficient Conditions for Implementability

Next, I proceed to find conditions on model primitives under which an implementable commitment outcome must exist. The main results are stated below; to simplify notation, I write v_i instead of $v(a_i)$.

Proposition 3.7. *Let $h(\gamma_i; \gamma_{i+1})$ solve $\mathbb{E}[\omega \mid \omega \in [h(\gamma_i; \gamma_{i+1}), \gamma_{i+1}]] = \gamma_i$.*[19] *If*

$$\frac{v_{i+1} - v_i}{\gamma_{i+1} - \gamma_i} > \frac{v_i - v_{i-1}}{\min\{\gamma_i - \gamma_{i-1}, \gamma_i - h(\gamma_i; \gamma_{i+1})\}}, \tag{2}$$

for all $i = 1, \ldots, n-2$, then every bi-pooling solution can be implemented. Consequently, the commitment payoff is attained in an equilibrium of the game.

Proposition 3.7 states that if it is sufficiently more profitable for Sender to induce the higher action than the lower one for any pair of adjacent actions, then she does not benefit from commitment power. How much more profitable is "sufficient" is jointly determined by the prior distribution and the position of the cutoffs. Loosely speaking, Condition (2) is satisfied whenever (i) the linear interpolation of the jumps in Sender's value function is convex, and (ii) the prior distribution does not accumulate too much mass around any point below the highest cutoff.

To get a closer look into how Condition (2) works, consider the following example. Let $\{B_i\}_{i=0}^{n-1}$ be a canonical representation of a bi-pooling solution, and suppose B_3 is a

[19] If $\mathbb{E}[\omega \mid \omega \in [h, \gamma_{i+1}]] < \gamma_i$ for all $h \geq 0$, set $h = 0$.

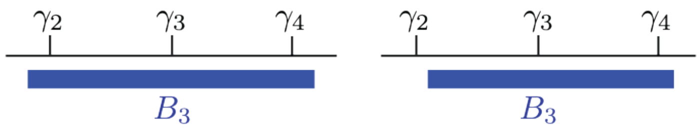

Figure 4: The two cases when B_3 is a pooling interval with $\bar{b}_3 > \gamma_4$. In the left panel there exists $\gamma_k \in B_3$ with $k < 3$, and in the right panel $\gamma_2 \notin B_3$.

pooling interval with $\bar{b}_3 > \gamma_4$.[20] By Proposition 3.5, this bi-pooling solution cannot be implemented; I show that Sender can strictly improve her payoff, which leads to a contradiction.

There are two cases: there exists $\gamma_k \in B_3$ with $k < 3$, or there is not; these two cases are illustrated in Figure 4. In the first case, without loss of generality, assume $\gamma_2 \in B_3$; and in the second case, it must be that $\gamma_2 < \underline{b}_3$. Observe that Condition (2) is equivalent to that both of the inequalities below must hold:

$$\frac{v_{i+1} - v_i}{\gamma_{i+1} - \gamma_i} > \frac{v_i - v_{i-1}}{\gamma_i - \gamma_{i-1}}, \tag{3}$$

$$\frac{v_{i+1} - v_i}{\gamma_{i+1} - \gamma_i} > \frac{v_i - v_{i-1}}{\gamma_i - h\left(\gamma_i; \gamma_{i+1}\right)}. \tag{4}$$

Interpolating (γ_0, v_0), (γ_1, v_1), ..., and (γ_{n-1}, v_{n-1}), a piecewise linear function on $[0, \gamma_{n-1}]$ is obtained; illustrated in the left panel of Figure 5, inequality (3) says that the slope of any "piece" is strictly larger than the slope of the piece to its left. Then in the first case, as shown in the right panel of Figure 5, (3) implies that splitting some mass on γ_3 to γ_2 and γ_4 is strictly profitable. In the proof, I show that such a profitable deviation is feasible, in the sense that the resulting signal distribution is still feasible for the information design problem (1). In the second case where $\gamma_2 \notin B_3$, $h(\gamma_3; \gamma_4)$, which is the point that the conditional mean of the states between this point and γ_4 is exactly γ_3, must belong to B_3 since $\gamma_4 \in B_3$. Replacing γ_2 by $h(\gamma_3; \gamma_4)$, by inequality (4), the previous argument goes through *mutatis mutandis.*

Imposing an assumption on the prior, Condition (2) can be substantially simplified.

Corollary 3.8. *If f is increasing,*[21] $v_{i+1} - v_i \geq v_i - v_{i-1}$, *and* $\gamma_{i+1} - \gamma_i \leq \gamma_i - \gamma_{i-1}$, *with one of the inequalities being strict for all $i = 1, \ldots, n-1$, then every bi-pooling solution can be implemented. Consequently, the commitment payoff is attained in an equilibrium.*

[20] If instead B_3 is a component of a nested intervals representation, and $B_3 = [\underline{b}_3, \bar{b}_3]$, the same logic goes through; if $B_3 = [\underline{b}_3, l] \cup [h, \bar{b}_3]$, some more work is needed but the underlying idea is similar.

[21] I use "increasing" and "greater" in the weak sense: "strictly" will be added whenever needed.

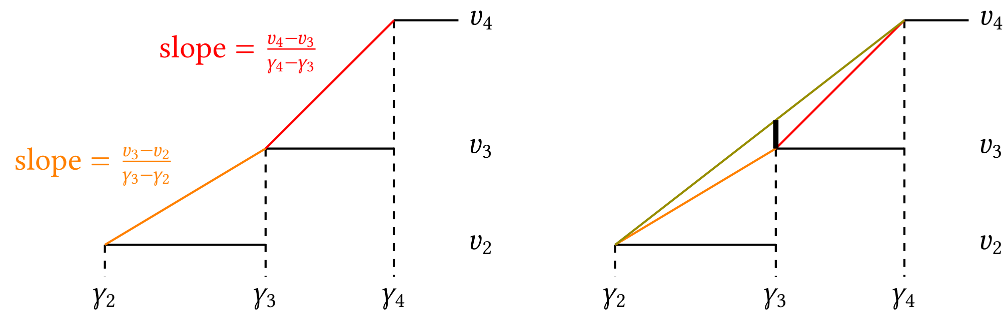

Figure 5: The left panel illustrates the geometric interpretation of inequality (3). The right panel shows that, when inequality (3) holds and $\gamma_2 \in B_3$, induce actions a_2 and a_4 more often, and action a_3 less often is strictly profitable to Sender.

When there are only three actions, the sufficient conditions can be further simplified. The proofs of all results concerning the ternary actions environment are relegated to the Supplementary Appendix.

Corollary 3.9. *If $|\mathcal{A}| = 3$, f is increasing, and $v_2 > 2v_1$, then all commitment outcomes can be implemented.*

Example 1 revisited. With the help of Corollary 3.9, to assert that Sender does not benefit from commitment power, one does not need to solve the information design problem. Because the prior F is uniform, its density $f = 1$ is constant, and hence increasing. Then since $v_2 = 3v_1 > 2v_1$, by Corollary 3.9, the commitment payoff must be attained in an equilibrium of the verifiable disclosure game.

4 Obedient Recommendation Equilibria

In the class of verifiable disclosure games I study, if any of the sufficient conditions identified in Section 3.4 holds, there exists an equilibrium outcome of this game that coincides with a commitment outcome. As a consequence, to solve for the Sender's preferred equilibrium outcome, it suffices to solve the corresponding information design problem. Even if none of the sufficient conditions is met, one can nonetheless find a bi-pooling solution to the information design problem and use Proposition 3.5 to check whether it is implementable.

But if the above conditions are not satisfied, there might be no equilibrium that yields the commitment payoff for Sender. Consequently, one cannot rely on the information design problem in finding Sender's preferred equilibrium. The first main result in this section establishes that Sender's preferred equilibrium exists even if no commitment solution is implementable, and characterizes Sender's preferred equilibrium in this case. To arrive there, I first define the concept of obedient recommendation equilibria. The second main result concerns Sender's equilibrium payoff set: I show that any payoff sandwiched between the unraveling payoff and Sender's preferred equilibrium payoff can be attained in an obedient recommendation equilibrium.

To avoid making my notation system even more complicated, I abuse notation and call a collection of closed subsets of $[0,1]$, $\{B_i\}_{i=0}^{n-1}$, a deterministic representation if $\cup_{i=0}^{n-1} B_i = [0,1]$, and for any $i, j = 0, 1, \ldots, n-1$, $\mu_F\left(B_i \cap B_j\right) = 0$.

Definition 3. An equilibrium (σ, τ, p) of the verifiable disclosure game is an **obedient recommendation equilibrium** (ORE) if there exist a deterministic representation $\{B_i\}_{i=0}^{n-1}$ such that

(a) for each $i = 0, 1, \ldots, n-1$, $\omega \in B_i \setminus \cup_{j>i} B_j$ implies $\sigma(B_i \,|\, \omega) = 1$;

(b) for each $i = 0, 1, \ldots, n-1$, $\tau(a_i \,|\, B_i) = 1$.

In an ORE, in any state a message $m \in \{B_i\}_{i=0}^{n-1}$ is sent with probability one; and upon receiving message B_i, Receiver plays action a_i with probability one. Consequently, a message B_i in an ORE can be interpreted as a recommendation of action a_i, and Receiver finds it optimal to follow.

Say that a deterministic representation $\{B_i\}_{i=0}^{n-1}$ satisfies **obedience** if $\mathbb{E}[\omega \mid \omega \in B_i] \in A_i$ for each $i = 0, 1, \ldots, n-1$. That is, for each i, it is optimal for Receiver to play a_i when she believes that the state is contained in B_i. Lemma 4.1 characterizes the set of deterministic representations that are associated with some ORE. This result highlights an appealing property of this class of equilibria: finding an equilibrium can be reduced to finding a partition of the state space satisfying two properties.

Lemma 4.1. *A deterministic representation $\{B_i\}_{i=0}^{n-1}$ is associated with an ORE if and only if it is both obedient and incentive compatible.*

The "only if" direction of Lemma 4.1 follows directly from the definition of ORE. The idea behind the "if" direction is very similar to the second step in proving Proposition 3.4: the only difference is that there is no need to require obedience in the latter since a deterministic representation of a bi-pooling solution is by construction obedient.Here, obedience guarantees that using action a_i with probability 1 is a best response to B_i.

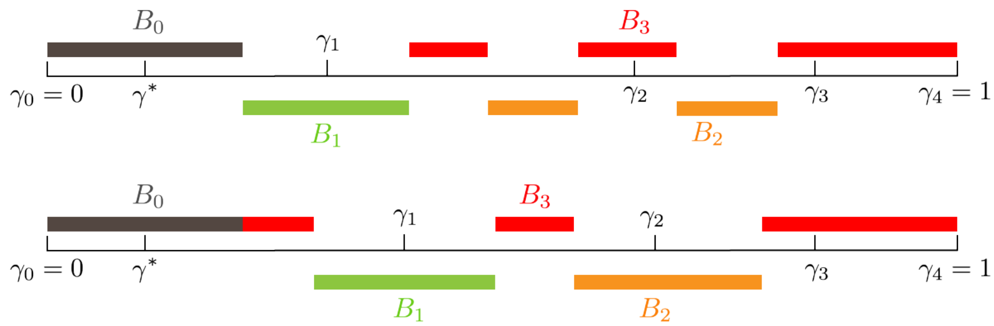

Figure 6: The deterministic representation in the upper panel is not a laminar representation since (5) fails to hold for B_3. The one in the lower panel is a laminar representation: (5) holds for each $i = 0, 1, 2, 3$; and for any pair of its elements, the convex hulls are either nested or have disjoint interiors.

4.1 Sender's Preferred Equilibrium

Say that a deterministic representation $\{B_i\}_{i=0}^{n-1}$ is a **laminar representation** if it satisfies[22]

$$B_i = \text{cl}\left(\text{co}\,(B_i) \Big\backslash \bigcup_{k<i} \text{co}\,(B_k)\right) \tag{5}$$

for each $i \in \{0, 1, \ldots, n-1\}$.[23] The definition implies that each element of the laminar representation can be identified by the smallest closed interval that contains it, and it is obtained by "taking out" the convex hulls of the elements with a lower index number from the aforementioned closed interval.

Directly, if $\{B_i\}_{i=0}^{n-1}$ is a laminar representation with $i < j$, either $\text{int}\,(\text{co}\,(B_i)) \cap \text{int}\left(\text{co}\left(B_j\right)\right) = \emptyset$, or $\text{co}\,(B_i) \cap \text{co}\left(B_j\right) = \text{co}\,(B_i)$. In words, for any two elements of the laminar representation, either the interiors of their respective convex hulls do not intersect, or the convex hull of the element with a higher index contains the one with a lower index. Furthermore, let $j = \min\{i : \text{int}\,(B_i) \neq \emptyset\}$, by definition it must be that $B_j = \text{co}\,(B_j)$, and hence B_j must be an interval. It is straightforward to see that a canonical representation is laminar. Figure 6 illustrates the restrictions imposed by the laminar structure.[24]

[22] For a subset S of $[0, 1]$, let $\text{cl}(S)$ and $\text{co}\,(S)$ denote its closure and convex hull, respectively.

[23] The term "laminar" is inspired by Candogan and Strack (2022). In fact, if $\{B_i\}_{i=0}^{n-1}$ is a laminar representation, and for each $i = 0, \ldots, n-1$, let $J_i = B_i \backslash \cup_{k<j} B_k$, then $\{J_i\}_{i=0}^{n-1}$ is a laminar partition according to Definition 2 in Candogan and Strack (2022). However, my definition of laminar representation is less permissive than their definition of laminar partition.

[24] As further examples, the deterministic representation in the upper panel of Figure 2 is not a laminar representation, and the one in the lower panel is a laminar representation because it is a canonical representation.

Now I am ready to characterize Sender's preferred equilibrium.

Proposition 4.2. *There exists a Sender's preferred equilibrium which is an ORE such that*

(i) the associated deterministic representation $\{B_i\}_{i=0}^{n-1}$ is a laminar representation which satisfies

a. let $j = \min\{i : \text{int}(B_i) \neq \emptyset\}$, then B_j is an interval, and for each B_k with $k > j$, it is the union of at most $n-1$ disjoint intervals; and

b. $\mathbb{E}[\omega \mid \omega \in B_j] \in [\gamma_j, \gamma_{j+1})$ and $\mathbb{E}[\omega \mid \omega \in B_k] = \gamma_k$ for all k with $k > j$ such that a_k is not skipped; and

(ii) Sender's ex ante payoff is strictly higher than in any equilibrium that features unraveling.

Proposition 4.2 shows that there exists a Sender's preferred equilibrium that is an ORE associated with a laminar representation. Furthermore, every on-path message is the union of at most $n-1$ disjoint intervals. This implies that even if no commitment outcome can be implemented, Sender can rely on messages with a relatively simple structure to attain the highest equilibrium payoff. In particular, arbitrarily complex messages (for example, a message that is a disjoint union of countably many subsets of states) and/or mixed strategies are not needed. Moreover, Sender must maximally exploit the obedience constraint in this equilibrium, and she strictly prefers this equilibrium to an unraveling equilibrium.

To establish Proposition 4.2, I first show that for every Sender's preferred equilibrium, I can construct an ORE in which Sender's ex ante payoff is the same. To show that it suffices to focus on laminar representations, I first observe that an ORE induces a distribution over posterior means G with support on at most n points, and this distribution must be a MPC of the prior. Making use of a slight variant of a result in Candogan and Strack (2022), I show that such a distribution induces a laminar representation. The argument is completed by noting that a laminar representation is the most "deviation-proof" deterministic representation: if a deterministic representation is incentive compatible, there must exist a laminar representation that does the same. This is not too surprising since a laminar representation is a natural generalization of a canonical representation.

Now Lemma 4.1 implies that the search for Sender's preferred equilibrium can be reduced to finding a laminar representation that is both obedient and incentive compatible, and yields the highest Sender's ex ante payoff. To show that such a laminar representation exists, I first note that the space $\mathcal{C}_c$ of nonempty, closed, and convex subsets of

$[0,1]$ endowed with the Hausdorff distance is a compact metric space. Then since every component of a laminar representation is the union of at most $n-1$ closed intervals, it can be identified by an element of $\mathcal{C}_c^{n(n-1)}$;[25] $\mathcal{C}_c^{n(n-1)}$ is compact in the product topology. I show that (i) Sender's ex ante payoff is continuous on $\mathcal{C}_c^{n(n-1)}$, and (ii) the set of laminar representations that satisfy obedience and incentive compatibility is a nonempty closed subset of $\mathcal{C}_c^{n(n-1)}$. Then by the extreme value theorem, the desired laminar representation must exist; this begets the existence of Sender's preferred equilibrium.

The result that Sender's ex ante payoff in her preferred equilibrium is strictly higher than the unraveling payoff is intuitive because she can exploit Receiver's indifference when there are finitely many actions. To see that Sender must maximally exploit the obedience constraint, suppose $\mathbb{E}[\omega \mid \omega \in B_k] > \gamma_k$ for some $k > j = \min\{i : \operatorname{int}(B_i) \neq \emptyset\}$, then Sender can strictly increase her ex ante payoff by "taking out" a subset of B_j with strictly positive measure and "merge" it with B_k.

4.1.1 Three Actions

As an example, in this paragraph I solve for a Sender's preferred ORE when Receiver has three actions. In this case, a deterministic representation can be written as $\{B_0, B_1, B_2\}$; moreover, a laminar representation must be a canonical representation. Then by Proposition 4.2, to search for Sender's preferred equilibria, it suffices to restrict attention to canonical representations.

Claim 4.3. *A Sender's preferred equilibrium is an ORE defined by a canonical representation* $\{B_0, B_1, B_2\}$.

Armed with Claim 4.3, the Sender's preferred ORE can be explicitly solved.

Claim 4.4. *The Sender's preferred equilibrium outcome either coincides with the commitment outcome, or it can be induced by an ORE defined by* $\{B_0, B_1, B_2\}$ *such that* $B_2 = [y, h] \cup [\gamma_2, 1]$, $B_1 = [h, \gamma_2]$, *and* $B_0 = [0, y]$, *where* $h > 0$, $y \geq 0$ *solve the system of equations*

$$\begin{aligned} \mathbb{E}\left[\omega \mid \omega \in [h, \gamma_2]\right] &= \gamma_1 \\ \mathbb{E}\left[\omega \mid \omega \in [y, h] \cup [\gamma_2, 1]\right] &= \gamma_2 \end{aligned}.$$

When there are only three actions, the only reason that makes a commitment outcome not implementable is that the "middle action" a_1 is recommended too often. Therefore, in

[25] $\mathcal{C}_c^{n(n-1)}$ denotes the $n(n-1)$-fold Cartesian product of $\mathcal{C}_c$ with itself.

the canonical representation that corresponds to a Sender's preferred ORE, a_1 is recommended as frequently as incentive compatibility allows. More precisely, it must be that $\sup B_1 = \gamma_2$.

In fact, this observation holds more generally: if no commitment outcome is implementable, at least one of the intermediate actions is recommended as often as (IC) allows.[26] Put differently, there must exist at least one unskipped action a_i is such that $\bar{b}_i = \gamma_{i+1}$, where $\bar{b}_i = \sup B_i$.[27]

4.2 Other ORE and Equilibrium Payoff Set

Lemma 4.1 and Proposition 4.2 allow me to characterize Sender's equilibrium payoff set.

Proposition 4.5. *Sender can attain an ex ante payoff in an ORE if and only if this payoff is less than the payoff in her preferred equilibrium and is greater than the unraveling payoff.*[28] *Furthermore, any such payoff can be attained by a laminar representation.*

Proposition 4.5 shows that there is a continuum of obedient recommendation equilibria. By definition, in each of these equilibria, both Sender and Receiver play pure strategies; in other words, none of them are obtained through mixing. In particular, for distinct equilibrium payoffs, I can find different sets of on-path messages to support them as OREs. Given this "richness" property, it might not be too surprising that a collection of on-path messages which corresponds to an ORE is "focal", in the sense that both Sender and Receiver find it customary: Receiver understands how Sender pools the states into messages, and Sender knows what action Receiver would take upon seeing each of the on-path messages. If this is the case, an equilibrium outcome that differs from unraveling could be observed in the real life.

The "only if" direction of Proposition 4.5 is immediate: by definition, Sender's ex ante payoff in an ORE is lower than in her preferred equilibrium; and incentive compatibility implies that it must be higher than her payoff in an unraveling equilibrium. The proof

[26] By "intermediate actions" I mean the actions that are neither the highest nor the lowest.

[27] To see this, note first that it must be that $\bar{b}_i \le \gamma_{i+1}$ for each i such that a_i is not skipped: if $\bar{b}_k > \gamma_{k+1}$ for some unskipped a_k, $B_k \cap \operatorname{int}(A_{k+1}) \neq \emptyset$, and hence Condition (IC) is violated. Now it only suffices to rule out the possibility that $\bar{b}_i < \gamma_{i+1}$ for each i such that a_i is not skipped. If this is the case, since no commitment outcome is implementable, it is strictly more profitable for Sender to recommend an intermediate action more often. A contradiction.

[28] Recall that in any equilibrium that features unraveling, in each state $\omega \in [0,1]$, Sender sends the message $m = \{\omega\}$; put differently, Sender fully reveals the state. Directly, Sender's ex ante payoff is the same in all such equilibria; call this payoff the unraveling payoff.

of the other direction is constructive. Observe that the ORE defined by the deterministic representation $\{A_i\}_{i=0}^{n-1}$ generates the unraveling payoff R_U for Sender. Let $\{B_i^S\}_{i=0}^{n-1}$ denote the laminar representation associated with a Sender's preferred equilibrium with ex ante payoff R_S. Then for each $R \in [R_U, R_S]$, I find a laminar representation that is a "mixture" between $\{A_i\}_{i=0}^{n-1}$ and $\{B_i^S\}_{i=0}^{n-1}$ with ex ante payoff R; then I show that it is both obedient and incentive compatible. Thus, by Lemma 4.1, R can be induced by an ORE defined by this laminar representation. Consequently, for every $R \in [R_U, R_S]$, there exists an ORE in which Sender's ex ante payoff is R, and every on-path message is the union of at most $n-1$ intervals.

5 Applications

5.1 Selling with Verifiable Disclosure

In this paragraph, I study a variant of the model of a sales encounter studied in Section 5 of Milgrom (1981). The state of the world, ω, is interpreted as the quality of the seller's product. Let $p > 0$ be the unit price; assume that it is the "no-haggle" price and there is no quantity discount. Denote the seller's constant unit cost by c, where $0 \leq c < p$. The buyer is only allowed to buy integer units of the product, and her payoff from purchasing q units is $\omega U(q) - pq$, where $U : \mathbb{R}_+ \to \mathbb{R}$ is a bounded, strictly increasing, strictly concave threetimes differentiable function with $U(0) = 0$. I further assume that $U(q) - pq$ is maximized at $n > 1$. As a consequence, the buyer buys at most n units of the product, and she buys nothing if ω is close enough to 0.

The only significant difference between this model and Milgrom's is that he allows the buyer to choose noninteger quantities, and hence the seller's value function is strictly increasing. In this model; however, the restriction on integer units makes the seller's value function a step function with n jumps. Under his assumptions, Milgrom shows that every equilibrium of the game features full disclosure: the seller sends $m = \{\omega\}$ for each $\omega \in [0, 1]$, resulting in the buyer's preferred outcome. With the restriction on integer units; however, the seller may be able to attain her highest attainable payoff, namely the commitment payoff.

To state the result, let

$$A(x) = -\frac{U''(x)}{U'(x)}, \quad \text{and} \quad P(x) = -\frac{U'''(x)}{U''(x)}$$

denote the coefficients of absolute risk aversion and absolute prudence, respectively.

Claim 5.1. *If f is increasing and $P > 2A$, there exists an equilibrium of this game in which the seller is as well off as having commitment power.*

The proofs of results in this section and the next are left to the Supplementary Appendix. The assumption of increasing prior density can be interpreted as it is common knowledge that the consumer is "relatively confident" about the quality of the product. $P > 2A$ is satisfied by, for example, CRRA utility function with parameter $\sigma < 1$.

5.2 Influencing Voters

Consider an amendment voting setting where a voting rule satisfying the Condorcet winner criterion is employed to determine which one of the three alternatives, the (unamended) bill (b), the amended bill (ab), and no bill (or status quo; $\emptyset$), would win.[29] The state of the world is $\omega \in [0, 1]$, and there are $N > 1$ voters; for simplicity, assume that N is an odd number. Voters have linear preferences: for $j = 1, \dots, N$, voter j's utilities are given by $u_k^j(\omega) = \alpha_k^j + \beta_k^j \omega$, where $k \in \{b, ab, \emptyset\}$ and where $\omega \in [0, 1]$ is the state of the world. Moreover, $\beta_b^j > \beta_{ab}^j > \beta_\emptyset^j = 0$ and $0 = \alpha_\emptyset^j > \alpha_{ab}^j > \alpha_b^j$ for all $j = 1, \dots, N$. Let γ_1^j and γ_2^j denote the cutoff states that voter j is indifferent between $\emptyset$ and ab, and ab and b, respectively.[30] I impose further assumptions so that $\gamma_1^j < \gamma_2^j$ for all $j = 1, \dots, N$.

In this model, voter j's preferences over alternative k are characterized by two parameters: one is α_k^j, I call it "reference point" as it is the voter's cardinal utility when the state is zero; another is β_k^j, I call it "state sensitivity" because it measures how fast the voter's cardinal utility increases in the state. Furthermore, all voters agree on that when the state is low (intermediate, high), no bill (the amended bill, the bill, respectively) is optimal; but for the amended bill or the bill, different voters may have different reference points and different state sensitivity levels. Figure 7 illustrates a voter's utilities and the resulting cutoffs.

There is an expert who observes the state and can communicate to the voters. I follow Jackson and Tan (2013) to assume that the expert discloses verifiable information. Unlike their work with two states; however, I consider a continuum of states, and the expert's

[29] See, for example, Enelow and Koehler (1980) and Enelow (1981) for examples of amendment voting.

[30] For every $j = 1, \dots, N$,

$$\gamma_1^j = -\frac{\alpha_{ab}^j}{\beta_{ab}^j} \quad \text{and} \quad \gamma_2^j = \frac{\alpha_{ab}^j - \alpha_b^j}{\beta_b^j - \beta_{ab}^j}.$$

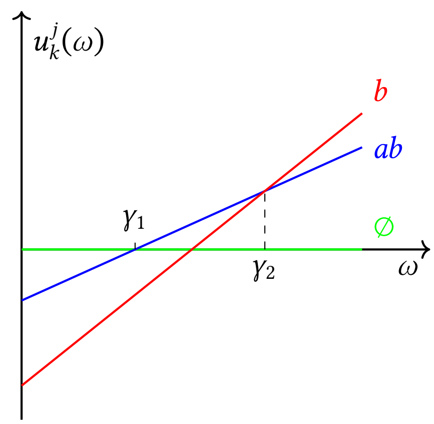

Figure 7: Voter j's utilities.

messages are closed subsets of the state space that contains the true state. Assume that the expert's preferences satisfy $v(b) > v(ab) > v(\varnothing) = 0$; that is, the expert strictly prefers the bill to the amended bill, and the amended bill is strictly preferred to no bill.

Because the voting rule satisfies the Condorcet winner criterion, and the preferences are single-peaked, the Condorcet winner is the median voter's most preferred alternative.[31] Therefore, it suffices to consider the median voter; further assumptions are imposed to make sure that the median voter is the same voter, say voter m, for all states. Consequently, the expert's problem is equivalent to communicating to voter m.[32] Thus, the expert's preferred equilibrium of this game is characterized by Claim 4.4.

Perhaps surprisingly, Claim 5.2 shows that the expert can be hurt if all voters are "more inclined toward" the bill, in the sense that all else equal, α_b^j and/or β_b^j increase for all $j = 1, \dots, N$.

Claim 5.2. *If no commitment outcome is implementable, and at least one of the following happens:*

(i) β_b^j *increases for all* $j = 1, \dots, N$,

(ii) α_b^j *increases for all* $j = 1, \dots, N$,

[31] A voting rule used for amendment voting that satisfies the Condorcet winner criterion when voters' preferences are single-peaked is the pairwise majority rule, in which alternatives will be considered sequentially and two at a time using the majority rule.

[32] That is, I assume that for all $j < i$, $\gamma_1^j \leq \gamma_1^j$ and $\gamma_2^j < \gamma_2^j$ (see Footnote 30 for the definition of the cutoffs). A sufficient condition for this is that $\alpha_b^j := \alpha_b$, $\alpha_{ab}^j := \alpha_{ab}$ for all $j = 1, \dots, N$; and for any $i > j$, $\alpha_{ab}^i > \alpha_{ab}^j$, and $\beta_b^i - \beta_{ab}^i > \beta_b^j - \beta_{ab}^j$. This assumption says that (i) all voters have the same reference points for all the alternatives; (ii) a higher indexed voter has a higher state sensitivity for both b and ab, and the difference between the state sensitivities for b and ab are larger for a voter with a higher index.

then the expert's payoff in her preferred equilibrium may decrease.

When either (i) or (ii) happens, or both, γ_2^j decreases for all j, and hence γ_2^m must decrease. Recall from the discussion after Claim 4.4 that, when no commitment outcome is implementable, it must be that the expert is too tempted to recommend the amended bill at the ex ante stage. This is because the payoff gap between the bill and the amended bill is not as large as the counterpart between the amended bill and the status quo, which is likely in many voting scenarios. Then as γ_2^m falls, although the expert can induce the bill more frequently, there is also an indirect effect that incentive compatibility becomes "tighter", and hence the amended bill is induced less often. The expert is hurt by this change if the indirect effect dominates.

6 Extensions

In this section, I discuss two extensions of the basic model. The first one, which concerns a multidimensional state space, is relegated to the Supplementary Appendix. There I show that Proposition 3.4 and Corollary 3.6 naturally extend.

Another direction to extend the baseline model is that the state is still unidimensional, and the sender's preferences can be more general. I keep the assumption that Sender's payoff only depends on Receiver's action, but instead of an increasing step function, I only impose the following regularity conditions on Sender's value function:

Definition 4. A function $u : [0,1] \to \mathbb{R}$ is **regular** if

(i) (Dizdar and Kováč, 2020) u is bounded, upper semicontinuous, and there exists $\varepsilon > 0$ such that it is Lipschitz continuous on $[0, \varepsilon]$ and $[1-\varepsilon, 1]$;

(ii) u is either lower semicontinuous or monotone.[33]

In particular, I allow Receiver's action space to be a continuum. It is also not hard to see that the increasing step function with finite jump points considered in the baseline model is regular. Proposition 6.1 characterizes implementability in this more general environment.

Proposition 6.1. *A bi-pooling solution G_B is implementable if and only if for every bi-pooling interval $[\underline{\omega}_i, \overline{\omega}_i]$ where* $\text{supp}\left(G_B|_{[\underline{\omega}_i, \overline{\omega}_i]}\right) = \{z_L^i, z_H^i\}$,[34] *there exists a bi-partition $\{B_L^i, B_H^i\}$*

[33] A function $u : [0,1] \to \mathbb{R}$ is monotone if it is either increasing or decreasing.

[34] A pooling interval is a special case of a bi-pooling interval with $z_L^i = z_H^i$.

such that[35]

$$u(z_L^i) = \max_{x \in B_L^i} u(x), \text{ and } u(z_H^i) = \max_{x \in B_H^i} u(x). \tag{6}$$

The proof of Proposition 6.1 proceeds similarly to the proof of Proposition 3.4; the only key difference is that a more general notion of maximally skeptical beliefs is needed. For any closed subset m of $[0, 1]$, let $S_m = \arg\min_{x \in m} u(x)$;[36] then for any off-path message m, let $p(S_m \mid m) = 1$. The maximally skeptical beliefs I defined after Proposition 3.4 is a special case: in the baseline model $\min m \in S_m$.

There are two reasons for which I focus on bi-pooling solutions. First is because Theorem 3.1 still holds in this more general environment, and hence every information design problem admits a bi-pooling solution. Secondly, a bi-pooling solution can be identified using the duality approach studied in Dworczak and Martini (2019) and Section 4 of Arieli et al. (2022). In particular, if there exist $k \in \mathbb{N}$ and a sequence of points $0 = y_0 < y_1 < \cdots < y_k = 1$ such that Sender's value function u is either concave or concave on (y_{i-1}, y_i) for every $i = 1, \ldots, k$, according to Proposition 3 in Arieli et al. (2022), there are at most $2k - 1$ bi-pooling intervals, so implementability boils down to checking condition (6) at most $2k - 1$ times. Furthermore, for simple value functions, a bi-pooling solution can be solved with a sheet of paper and a pencil.[37]

That being said, determining whether there exists a bi-partition of some bi-pooling intervals that satisfy (6) can be hard in some cases. For some special classes of value functions; however, it suffices to check some simpler conditions.

Corollary 6.2. *If u is differentiable, a bi-pooling solution G_B is implementable if and only if for every bi-pooling interval $[\underline{\omega}_i, \overline{\omega}_i]$ with* $\operatorname{supp}\left(G_B|_{[\underline{\omega}_i, \overline{\omega}_i]}\right) = \{z_L^i, z_H^i\}$ *satisfies*

$$u(z_L^i) = u(z_H^i) = \max_{x \in [\underline{\omega}_i, \overline{\omega}_i]} u(x). \tag{7}$$

Observe that (7) is a very strong condition: for example, if u is generic in the sense that no two peaks admit the same value, and there is a bi-pooling interval in the bi-pooling solution, then u not implementable.[38] Even if the bi-pooling solution features no bi-pooling intervals (that is, pooling intervals only), the mean of *each* of the pooling intervals must

[35] The use of the maximum operator is justified since both B_L^i and B_H^i are closed subsets of a compact interval, hence themselves compact, and u is assumed to be upper semicontinuous.

[36] S_m is nonempty by Condition (ii) of regularity.

[37] See Section 4 in Arieli et al. (2022) for two such examples.

[38] For readers who are familiar with Dworczak and Martini (2019), the condition in Corollary 6.2 is equivalent to that their "price function" has to be flat on each of the pooling and bi-pooling intervals.

coincide with the point that maximizes Sender's value function on that interval, which is unlikely. Consequently, when Sender's value function is differentiable, she generically benefits from commitment power.

Corollary 6.3. *If u is monotone, a bi-pooling solution G_B is implementable if and only if for every bi-pooling interval $[\underline{\omega}_i, \overline{\omega}_i]$ with $\operatorname{supp}\left(G_B|_{[\underline{\omega}_i,\overline{\omega}_i]}\right) = \{z_L^i, z_H^i\}$, its nested intervals representation $\{B_L, B_H\}$ are such that (6) holds.*

Corollary 6.4 is a consequence of Corollary 6.2 and Corollary 6.3.

Corollary 6.4. *If u is convex, the signal distribution induced by full disclosure is a bi-pooling solution, and hence implementable. Otherwise, no commitment outcome is implementable if one of the following holds:*

(1) u is strictly monotone;

(2) u is monotone and differentiable.

An important implication of Corollary 6.4 is that if Sender's value function u is monotone and non-convex, both "jumps" and "flat parts" are necessary for implementation.

7 Discussion

7.1 Message Space

Although my assumption on the message space is identical to Grossman (1981) and Milgrom (1981), there are some alternative verifiability assumptions used in the literature. One is the "truth-or-nothing" message space: $\mathcal{M}(\omega) = \{\{\omega\}, [0, 1]\}$ (for example, Dye (1985) and Jung and Kwon (1988)); that is, Sender either fully discloses the state or just claims that "I have nothing to say". Another is the space of closed intervals: $\mathcal{M}(\omega) = \{m \subseteq [0, 1] : m \text{ is a closed interval, and } \omega \in m\}$ (for example, Hagenbach and Koessler (2017) and Ali et al. (2022)). The assumption on the message space is crucial to a majority of my results: Ali et al. (2022) call the former "simple evidence" and the latter "rich evidence"; however, even rich evidence might not be "rich enough" for implementation. This is because, so long as a commitment outcome features bi-pooling, a necessary condition for implementation is that Sender has to be able to send a message that is the union of two disjoint closed intervals.[39] When no commitment outcome is implementable, Sender may need to

[39]One may think one can implement a bi-pooling solution using "rich evidence" by identifying each message by its convex hull. To see why this does not work, suppose $[\underline{\omega}, \overline{\omega}]$ is bi-pooled to $\{\gamma_L, \gamma_H\}$. By

send a message that is the union of up to $n-1$ disjoint closed intervals in her preferred equilibrium. Evidently, the message space in my model is richer than "rich evidence".

7.2 Robustness

I show in the Supplementary Appendix that every obedient recommendation equilibrium of the game I study, after a slight adjustment in the belief system, survives the Never-a-Weak-Best-Response (NWBR) criterion proposed by Cho and Kreps (1987).[40] NWBR is a strengthening of the Cho-Kreps Criterion, D1 and D2 that are extensively used in the literature.

7.3 Cheap Talk

One may wonder what would Sender be able to achieve if she is only allowed to send cheap talk messages. This question can be answered by appealing to Theorem 2 in Lipnowski and Ravid (2020). To use it; however, I need to define Sender's **value function in beliefs**, denoted by $w : \Delta([0,1]) \to \mathbb{R}$, by $w(G) = u(\mathbb{E}G)$ for any $G \in \Delta([0,1])$, where u is Sender's value function. The aforementioned theorem asserts that Sender's optimal value is given by the quasiconcave hull of her value function in beliefs evaluated at the prior F.[41] Observe that w is the composite function of u and the expectation operator $\mathbb{E}$. Then because u is increasing, and that an increasing transformation of an affine function is quasiconcave, $w = \bar{w}$. Thus, Sender never benefits from communication if she is restricted to sending cheap talk messages.

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Lemma 3.3, the unique nested interval representation $\{B_L, B_H\}$ is given by $B_L = [\underline{b}_L, \overline{b}_L]$ and $B_H = [\underline{\omega}, \underline{b}_L] \cup [\overline{b}_L, \underline{\omega}]$. If one specifies that message $\text{co}(B_H) = [\underline{\omega}, \overline{\omega}]$ is sent in every $\omega \in B_H$; however, this message is also available to every $\omega \in B_L$. Consequently, in every $\omega \in [\underline{\omega}, \overline{\omega}]$, the message $[\underline{\omega}, \overline{\omega}]$ is sent, which renders replicating the bi-pooling solution impossible.

[40] Although closely related, this is not exactly the same as the NWBR property proposed by Kohlberg and Mertens (1986).

[41] The quasiconcave hull of a function $w : \Delta([0,1]) \to \mathbb{R}$, denoted by $\bar{w}$, is the pointwise lowest quasiconcave and upper semicontinuous function that majorizes w.

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A Omitted Proofs

A.1 Proofs for Section 3

A.1.1 Proof of Lemma 3.3

The proof of Lemma 4 in Arieli et al. (2022) proves the existence part. To see uniqueness, let $\{B'_L, B'_H\}$ be a distinct nested interval representation where $B_L = [\underline{b}'_L, \overline{b}'_L]$ and $B_H = [\underline{\omega}, \underline{b}'_L] \cup [\overline{b}'_L, \overline{\omega}]$. Then either $\underline{b}'_L \neq \underline{b}_L$ or $\overline{b}'_L \neq \overline{b}_L$, or both. Since $f > 0$ and $\underline{\omega}$ and $\overline{\omega}$ are fixed, either $\mathbb{E}[B_L] \neq \gamma_L$, or $\mathbb{E}[B_H] \neq \gamma_H$, or both. A contradiction.

A.1.2 Proof of Proposition 3.4

I start by showing that if a bi-pooling solution is implementable, the equilibrium that induces it can be identified by a deterministic representation.

Lemma A.1. *If a bi-pooling solution G_B is implementable, there exists an equilibrium that induces G_B in which both players only use pure strategies. Consequently, in this equilibrium, every state maps to a single action with probability 1.*

Proof of Lemma A.1. Note first that two necessary conditions for a bi-pooling solution G_B to be implemented by an equilibrium (σ, τ, p) are

(I) if $(\underline{\omega}, \overline{\omega})$ is a pooling interval with barycenter γ, for any $\omega \in (\underline{\omega}, \overline{\omega})$ and $m \in \operatorname{supp} \sigma(\cdot \mid \omega)$, $m \subseteq (\underline{\omega}, \overline{\omega})$ and $\mathbb{E}[\omega \mid \omega \in m] = \gamma$;

(II) if $(\underline{\omega}, \overline{\omega})$ is a bi-pooling interval such that $\operatorname{supp}\left(G_B|_{(\underline{\omega}, \overline{\omega})}\right) = \{\gamma_L, \gamma_H\}$, for any $\omega \in (\underline{\omega}, \overline{\omega})$ and $m \in \operatorname{supp} \sigma(\cdot \mid \omega)$, $m \subseteq (\underline{\omega}, \overline{\omega})$, and either $\mathbb{E}[\omega \mid \omega \in m] = \gamma_L$ or $\mathbb{E}[\omega \mid \omega \in m] = \gamma_H$.

(I) and (II) are necessary because if any of them is violated, it is not possible for the bi-pooling solution G_B to be a part of an equilibrium outcome.

I first show that it is without loss to assume that to implement G_B, Sender plays pure strategies; that is, for every $\omega \in [0,1]$, supp $\sigma(\cdot\,|\,\omega)$ is a singleton. To see this, suppose an equilibrium (σ,τ,p) implements a bi-pooling solution G_B, and in some $\omega \in [0,1]$, $|\operatorname{supp}\sigma(\cdot\,|\,\omega)| > 1$. Because (σ,τ,p) is an equilibrium, and Receiver only mixes between adjacent actions, for every $m, m' \in \operatorname{supp}\sigma(\cdot\,|\,\omega)$, $\tau(\cdot\,|\,m) = \tau(\cdot\,|\,m')$. Furthermore, because (σ,τ,p) implements G_B, by conditions (I) and (II) above there must exist a unique $\gamma \in \{\gamma^*, \gamma_1, \dots, \gamma_{n-1}\}$ such that the posterior mean after receiving both messages is almost surely γ. Then let $\tilde{m}(\omega) = \cup \operatorname{supp}\sigma(\cdot\,|\,\omega)$, and define $\tilde{\sigma}$ by $\tilde{\sigma}(\tilde{m}(\omega)\,|\,\omega) = 1$ if $|\operatorname{supp}\sigma(\cdot\,|\,\omega)| > 1$ and $\tilde{\sigma}(\cdot\,|\,\omega) = \sigma(\cdot\,|\,\omega)$ otherwise, $(\tilde{\sigma},\tau,p)$ is an equilibrium that Sender only uses pure strategies, and it induces G_B since (σ,τ,p) does.

Since (σ,τ,p) is an equilibrium that induces G_B, it must be that $\tau(a_i\,|\,m) = 1$ for all m such that $\mathbb{E}[\omega\,|\,\omega \in m] = \gamma_i$; and if $\gamma^* \in [\gamma_j, \gamma_{j+1})$, $\tau(a_j\,|\,m) = 1$ for all m such that $\mathbb{E}[\omega\,|\,\omega \in m] = \gamma^*$. In words, Receiver must play pure strategies after receiving any message in the original equilibrium, and hence also in the equilibrium in which Sender only plays pure strategies. As a consequence, in every $\omega \in [0,1]$, a single action is played with probability 1 by Receiver; let W_i be the set of states in which action a_i is played with probability 1, and let B_i be the closure of W_i. Then it is straightforward to see that $\{B_i\}_{i=0}^{n-1}$ is a deterministic representation of G_B that is obtained in an equilibrium. ∎

Lemma A.2. *A bi-pooling solution is implementable if and only if there exists an incentive compatible deterministic representation* $\{B_i\}_{i=0}^{n-1}$ *of it.*

Proof of Lemma A.2. For the "only if" direction, suppose a bi-pooling solution G_B is implementable. By Lemma A.1, there is an equilibrium that induces G_B that can be identified by a deterministic representation. Suppose that (IC) does not hold for all deterministic representations of a bi-pooling solution. Then for any $\{B_i\}_{i=0}^{n-1}$ there exists $k \in \{1,\dots,n-1\}$ such that $\operatorname{int} A_k \cap B_\ell \neq \emptyset$ for some $\ell < k$. Consequently, it is strictly profitable for any $\omega \in \operatorname{int} A_k \cap B_\ell$ to deviate to $m = \{\omega\}$, which is not consistent with any bi-pooling solution. A contradiction.

For the other direction, suppose there exists an incentive compatible deterministic representation $\{B_i\}_{i=0}^{n-1}$ of a bi-pooling solution. I construct an equilibrium whose outcome coincides with the outcome induced by G_B. Consider Sender's strategy $\sigma : [0,1] \to \Delta(\mathcal{C})$ with $\operatorname{supp}\sigma(\cdot\,|\,\omega) \subseteq \{B_i\}_{i=0}^{n-1}$ for all $\omega \in [0,1]$ is defined by $\sigma(B_i\,|\,\omega) = 1$ if $\omega \in B_i$ and $\omega \notin B_j$ for all $j > i$, Receiver's strategy $\tau : \mathcal{C} \to \Delta(\mathcal{A})$ is defined by $\tau(a_i\,|\,B_i) = 1$ for all $i = 0, 1, \dots, n-1$, and for $m \notin \{B_i\}_{i=0}^{n-1}$, $\tau(a_i\,|\,m) > 0$ if and only if $\mathbb{E}[\omega\,|\,\omega \in$

$m] \in [\gamma_i, \gamma_{i+1}]$. Let the belief system be such that $p(m)$ is obtained using Bayes rule for all $m \in \{B_i\}_{i=0}^{n-1}$, and $p(\min m \,|\, m) = 1$ if $m \notin \{B_i\}_{i=0}^{n-1}$. By definition, action a_i is optimal to Receiver; and by incentive compatibility, in any state Sender does not want to deviate to full disclosure. Furthermore, the belief system guarantees that any other deviation is at most as good as full disclosure, and hence Sender never wants to deviate to any off-path message $m \notin \{B_i\}_{i=0}^{n-1}$ either. Consequently, (σ, τ, p) is an equilibrium. By construction, the outcome of this equilibrium is the same as the outcome induced by G_B. Therefore, G_B is implementable. ∎

Lemma A.3. *For every incentive compatible deterministic representation of a bi-pooling solution G_B, there exists a canonical representation of G_B that is also incentive compatible.*

Proof of Lemma A.3. Fix an incentive compatible deterministic representation $\{B_i^o\}_{i=0}^{n-1}$ of a bi-pooling solution G_B. Construct a canonical representation $\{B_i^c\}_{i=0}^{n-1}$ as follows: for every i such that B_i^o is a pooling interval, let $B_i^c = B_i^o$; and for every bi-pooling interval, say $B_j^o \cup B_k^o$, construct a nested intervals representation $\{B_j^c, B_k^c\}$ à la Lemma 3.3. By construction, for every $i = 0, \dots, n$, $\mu_F\left(B_i^o\right) = \mu_F\left(B_i^c\right)$, and $\mathbb{E}[\omega \mid \omega \in B_i^o] = \mathbb{E}[\omega \mid \omega \in B_i^c]$. If B_j^o is such that $\{B_j^o, B_k^o\}$ is a bi-partition with $j < k$, it must be that $B_j^o \cup B_k^o = B_j^c \cup B_k^c$; and if B_j^o is a pooling interval, $B_j^o = B_j^c$. Consequently, in these two cases, $\cup_{\ell \geq j} B_\ell^o = \cup_{\ell \geq j} B_\ell^c$, and thus $A_j \subseteq \cup_{\ell \geq j} B_\ell^o$ implies that $A_j \subseteq \cup_{\ell \geq j} B_\ell^c$.

Then to show that $\{B_i^c\}_{i=0}^{n-1}$ is also incentive compatible, it remains to consider k's such that $\{B_j^o, B_k^o\}$ is a bi-partition with $j < k$ and the skipped actions. For the first kind, it suffices to show that $A_k \subseteq \cup_{\ell \geq k} B_\ell^c$, which is equivalent to $\text{int}(A_k) \cap \left(\cup_{\ell < k} B_\ell^c\right) = \emptyset$, for every such k. I first claim that $\sup\left(\text{int}(B_j^o)\right) \geq \sup B_j^c$. To see this, suppose to the contrary that $\sup\left(\text{int}(B_j^o)\right) < \sup B_j^c$. Because $f > 0$, $\mu_F\left(B_i^o\right) = \mu_F\left(B_i^c\right)$, and that B_j^c is an interval, it must be that $\mathbb{E}[\omega \mid \omega \in B_j^o] < \mathbb{E}[\omega \mid \omega \in B_j^c]$. However, $\mathbb{E}[\omega \mid \omega \in B_i^o] = \mathbb{E}[\omega \mid \omega \in B_i^c]$ by construction; a contradiction. Because $A_k \subseteq \cup_{\ell \geq k} B_\ell^o$, it must be that $\text{int}(A_k) \cap \left(\cup_{\ell < k} B_\ell^o\right) = \emptyset$. Now note that $\cup_{\ell < k, \ell \neq j} B_\ell^o = \cup_{\ell < k, \ell \neq j} B_\ell^c$, and hence $\sup\left(\text{int}(B_j^o)\right) \geq \sup B_j^c$ would imply that $\text{int}(A_k) \cap \left(\cup_{\ell < k} B_\ell^c\right) = \emptyset$.

Finally, if a_m is skipped, $\cup_{\ell \geq m} B_\ell^o = \cup_{\ell \geq m} B_\ell^c$ unless $j < m < k$, where $\{B_j^o, B_k^o\}$ is a bi-partition of a bi-pooling interval. Hence, it suffices to consider this case only. Because $\{B_i^o\}_{i=0}^{n-1}$ is incentive compatible, $\text{int}(A_m) \cap \left(\cup_{\ell < m} B_\ell^o\right) = \emptyset$, and this implies that $A_m \subseteq B_k^o$: suppose not, then $\gamma_{m+1} > \sup B_k^o$, but this is impossible because $\gamma_{m+1} \leq \gamma_k < \sup B_k^o$. Then there are two cases: either $A_m \subseteq \text{co}\left(\text{int}(B_j^o)\right)$, or $\sup\left(\text{int}(B_j^o)\right) \leq \gamma_m$. The first case is not possible because $A_m \subseteq \text{co}\left(\text{int}(B_j^o)\right)$ implies that there exists $t > m > j$ such that

$\text{int}(A_t) \cap B_j^o \neq \emptyset$, which violates the assumption that $A_i \subseteq \left(\cup_{\ell<i} B_\ell^o\right)$ for all $i = 0, 1, \ldots, n-1$. For the second case, the argument in the previous paragraph, *mutatis mutandis*, shows that $\text{int}(A_m) \cap \left(\cup_{\ell<m} B_\ell^c\right) = \emptyset$.

Thus, the canonical representation $\{B_i^c\}_{i=0}^{n-1}$ is also incentive compatible. ■

Proposition 3.4 then follows from Lemma A.2 and Lemma A.3.

A.1.3 Proof of Proposition 3.5

Observe first that (IC) requires that if a_i is skipped, it must be skipped in a "harmless" way. An action a_i is **harmlessly skipped** in a canonical representation if it is skipped and $A_i \subseteq \cup_{k=i+1}^{n-1} B_k$. It is not hard to see that every action except a_0 is harmlessly skipped is a necessary condition for (IC): note that a_i is not harmlessly skipped if and only if $B_i = \emptyset$, and there exists B_k with $k < i$ such that $\text{int}(A_i) \cap B_k \neq \emptyset$. When $B_i = \emptyset$, $\cup_{k=i}^{n-1} B_k = \cup_{k=i+1}^{n-1} B_k$; so (IC) implies that for all $i = 1, \ldots, n-1$, if a_i skipped, it must be skipped harmlessly.

Lemma A.4 provides a characterization for an action to be harmlessly skipped.

Lemma A.4. *Let action a_i with $i \geq 1$ be skipped in a canonical representation $\{B_i\}_{i=0}^{n-1}$. Then it is harmlessly skipped if and only if for every unskipped action a_j with $j < i$, $\gamma_i \geq \underline{b}_j$.*

Proof of Lemma A.4. If a_i is skipped, and there exists $j < i$ with $B_j \neq \emptyset$ such that $\gamma_i < \overline{b}_j$, then $A_i \cap \text{int} B_j \neq \emptyset$, which implies that a_i is not harmlessly skipped. To prove the other direction, suppose a_i is skipped but not harmlessly skipped, I show that there exists $B_j \neq \emptyset$ with $j < i$ such that $\gamma_i < \overline{b}_j$. Because a_i is skipped but not harmlessly skipped, by definition there exists $s < i$ such that $A_i \cap \text{int} B_s \neq \emptyset$. Then there are two cases:

(1) $\gamma_i \in B_h$ for a pooling interval B_h with $h < i$, so $\gamma_i < \overline{b}_h$.

(2) $\gamma_i \in B_k \cup B_\ell = [\underline{\omega}, \overline{\omega}]$, where $\{B_k, B_\ell\}$ is a nested intervals representation with $k < \ell$ and $k < i$ ($i < k < \ell$ is not possible since in this case a_i cannot be harmlessly skipped). Recalling that $B_k = [\underline{b}_k, \overline{b}_k]$ and $B_\ell = [\underline{\omega}, \underline{b}_k] \cup [\overline{b}_k, \overline{\omega}]$, there are three possibilities:

  (2-1) $\gamma_i \in [\underline{\omega}, \underline{b}_k]$. Such case cannot emerge in a canonical representation: it must be that $\mathbb{E}[B_k] = \gamma_k$, then since $\gamma_i \in [\underline{\omega}, \underline{b}_k]$, $\gamma_i < \gamma_k$; but this is not possible since $i > k$.

  (2-2) $\gamma_i \in B_k$, so $\gamma_i < \overline{b}_k$.

  (2-3) $\gamma_i \in [\overline{b}_k, \overline{\omega}]$. If $k < i < \ell$, then a_i is harmlessly skipped; so it must be that $k < \ell < i$. Now note that $\gamma_i < \overline{\omega} = \overline{b}_\ell$.

The discussion above shows that there exists $B_j \neq \emptyset$ with $j < i$ such that $\gamma_i < \overline{b}_j$, which completes the proof. ■

Lemma A.5. *A canonical representation $\{B_i\}_{i=0}^{n-1}$ is incentive compatible if and only if*

- *for $i \geq 1$, if a_i is skipped, it must be harmlessly skipped;*
- *for any nested intervals representation $\{B_L, B_H\}$, $\bar{b}_L \leq \gamma_H$.*

Proof of Lemma A.5. Suppose the two conditions hold, I show that the canonical representation must satisfy (IC). I start from the "highest action", a_{n-1}. By definition, a_{n-1} cannot be harmlessly skipped, so it must be that it is never skipped, namely $B_{n-1} \neq \emptyset$. There are two cases: either B_{n-1} is a pooling interval, or $\{B_q, B_{n-1}\}$ is a nested intervals representation for some $q < n-1$. For the first case, know that $\mathbb{E}[\omega \,|\, \omega \in B_{n-1}] = \gamma_{n-1}$ and $\bar{b}_{n-1} = 1$, so $A_{n-1} = [\gamma_{n-1}, \gamma_n] \subseteq B_{n-1}$. For the second case, B_{n-1} must take the form of $[y, \underline{b}_q] \cup [\bar{b}_q, 1]$, and the second condition implies $\gamma_{n-1} \geq \bar{b}_q$; thus, $A_{n-1} \subseteq B_{n-1}$. Hence, in both cases, $A_{n-1} \subseteq B_{n-1}$.

Now consider a_{n-2}. If a_{n-2} is skipped, so $B_{n-2} = \emptyset$, then it must be that it is completely skipped, namely $A_{n-2} \subseteq B_{n-1} = B_{n-2} \cup B_{n-1}$. If instead $B_{n-2} \neq \emptyset$, there are three cases: (i) B_{n-2} is a pooling interval; (ii) $\{B_{n-2}, B_{n-1}\}$ is a nested intervals representation; (iii) $\{B_q, B_{n-2}\}$ is a nested intervals representation for some $q < n-2$. In case (i), $\mathbb{E}[\omega \,|\, \omega \in B_{n-2}] = \gamma_{n-2}$, and the only interval to the right of B_{n-2} is B_{n-1}, so $A_{n-2} \subseteq B_{n-2} \cup B_{n-1}$. In case (ii), $\mathbb{E}[\omega \,|\, \omega \in B_{n-2}] = \gamma_{n-2}$, and the only interval to the right of B_{n-2} is a component of B_{n-1}; thus, $A_{n-2} \subseteq B_{n-2} \cup B_{n-1}$. In case (iii), the second condition implies $\gamma_{n-2} \geq \bar{b}_q$, and the only interval to the right of B_{n-2} is B_{n-1}; thus, $A_{n-2} \subseteq B_{n-2} \cup B_{n-1}$. Therefore, in all three cases, $A_{n-2} \subseteq B_{n-2} \cup B_{n-1}$. Proceed in a similar manner, it can be shown that $A_i \subseteq \cup_{j \geq i} B_j$ for all i, which is exactly (IC).

Now suppose the canonical representation satisfies (IC). Then a necessary condition for (IC) to hold is that every skipped action a_i with $i \geq 1$ must be harmlessly skipped. For the second condition, suppose instead that $\gamma_H < \bar{b}_L$. This implies that $\text{int} A_H \cap B_L \neq \emptyset$, which violates (IC). Therefore, the second condition is also implied by (IC). This completes the proof. ∎

Proposition 3.5 then follows from Lemma A.4, Lemma A.5, and Proposition 3.4.

A.1.4 Proof of Corollary 3.6

I first claim that all bi-pooling solutions induce essentially the same canonical representation. If $\mathbb{E}_F[\omega] \geq \gamma_1$, action a_1 must be recommended with probability 1, and hence the essentially unique canonical representation has $B_1 = [0, 1]$, and $B_0 = \emptyset$. Suppose instead that $\mathbb{E}_F[\omega] < \gamma_1$, then any bi-pooling solution G_B must put mass $1 - F(x^*)$ at γ_1, where

x^* satisfies $\mathbb{E}[\omega \mid \omega \geq x^*] = \gamma_1$. Consequently, $\operatorname{supp} G_B \subseteq [0, x^*] \cup \{\gamma_1\}$, and the unique canonical representation has $B_1 = [x^*, 1]$, and $B_0 = [0, x^*]$.

Now it suffices to consider a single canonical representation. Clearly, a_1 cannot be skipped. Furthermore, $|\mathcal{A}| = 2$ implies that Sender's payoff as a function of posterior mean is a one-stepped function, so it is affine-closed (see Definition 2 in Dworczak and Martini (2019)). Theorem 3 in the same paper then implies that there does not exist a bi-pooling interval. Then by Proposition 3.5 there exists a bi-pooling solution that can be implemented.

A.1.5 Proof of Proposition 3.7

I use the following result due to Arieli et al. (2022) to prove Proposition 3.7. Say that $\{\gamma_L, \gamma_H\}$ **is a feasible bi-pooling support for the interval** $[\underline{\omega}, \overline{\omega}]$ (or just **feasible** for simplicity) if there exists a mean preserving contraction of $F|_{[\underline{\omega}, \overline{\omega}]}$ whose support is $\{\gamma_L, \gamma_H\}$.

Lemma A.6 (Arieli et al. 2022)**.** *Fix an interval $[\underline{\omega}, \overline{\omega}]$, and let $y(\gamma_L)$ satisfy $\mathbb{E}[\omega \mid \omega \in [\underline{\omega}, y(\gamma_L))] = \gamma_L$. Then $\{\gamma_L, \gamma_H\}$ is feasible for the interval $[\underline{\omega}, \overline{\omega}]$ if and only if*

(i) $\underline{\omega} \leq \gamma_L \leq \mathbb{E}[\omega \mid \omega \in [\underline{\omega}, \overline{\omega}]] \leq \gamma_H \leq \overline{\omega}$, *and*

(ii) $\mathbb{E}[\omega \mid \omega \in [y(\gamma_L), \overline{\omega}]] \geq \gamma_H$.

Proof of Proposition 3.7. Suppose to the contrary that there exists a bi-pooling solution G_B that is not implementable. Then by Proposition 3.5, the canonical representation of G_B, $\{B_i\}_{i=0}^{n-1}$, violates at least one of the two conditions in Proposition 3.5.

First suppose condition (i) is violated; that is, there exists a skipped action a_i, such that $\gamma_i < \overline{b}_j$ for an unskipped action a_j with $j < i$. Because a_j is not skipped, B_j must be nonempty. There are two cases.

Case 1. B_j is a pooling interval. There are two subcases:

(I) There exists $\gamma_k \in B_j$ with $k < j$.
For $z \in (\gamma_i, \overline{b}_j)$, let $h(z)$ solve $\mathbb{E}[\omega \mid \omega \in [\underline{b}_j, h(z)] \cup [z, \overline{b}_j]] = \gamma_i$. By Lemma A.6, for z close enough to $\overline{b}_j$, $\{\gamma_k, \gamma_j\}$ is feasible for $[h(z), z]$.Consequently, Sender's payoff on $[\underline{b}_j, \overline{b}_j]$, as a function of z, is

$$P(z) = v_i[F(\overline{b}_j) - F(z) + F(h(z)) - F(\underline{b}_j)] + \left[\frac{m(z) - \gamma_k}{\gamma_j - \gamma_k} v_j + \frac{\gamma_j - m(z)}{\gamma_j - \gamma_k} v_k\right] (F(z) - F(h(z)))$$

where $m(z) := \mathbb{E}[\omega \mid \omega \in [z, h(z)]]$. To show that this is a profitable deviation, it suffices to show that $P'(\bar{b}_j) := \lim_{z \nearrow \bar{b}_j} P'(z) < 0$. To this end, I first calculate

$$P'(z) = \frac{f(z)}{\gamma_j - \gamma_k} \frac{z - h(z)}{\gamma_i - h(z)} \left[(v_j - v_k)\gamma_i + (v_i - v_j)\gamma_k - (v_i - v_k)\gamma_j \right].$$

Letting $z \nearrow \bar{b}_j$, then $h(z) \searrow \underline{b}_j$, and

$$P'(\bar{b}_j) = \frac{f(\bar{b}_j)}{\gamma_j - \gamma_k} \frac{\bar{b}_j - \underline{b}_j}{\gamma_i - \underline{b}_j} \left[(v_j - v_k)\gamma_i + (v_i - v_j)\gamma_k - (v_i - v_k)\gamma_j \right];$$

clearly $P'(\bar{b}_j) < 0$ if and only if $(v_i - v_k)\gamma_j - (v_j - v_k)\gamma_i - (v_i - v_j)\gamma_k > 0$, and this is in turn equivalent to

$$\frac{v_i - v_j}{\gamma_i - \gamma_j} > \frac{v_j - v_k}{\gamma_j - \gamma_k},$$

which is implied by (2).

(II) There does not exist $\gamma_k \in B_j$ with $k < j$. In particular, it must be that $\gamma_{j-1} < \underline{b}_j$. WLOG, let $i = j + 1$.[42] Define $B_j^\varepsilon = [\underline{b}_j + \varepsilon, \bar{b}_j]$; for small enough $\varepsilon > 0$, $\{\gamma_j, \gamma_{j+1}\}$ is feasible for B_j^ε. Let $m(\varepsilon)$ denote the mean of B_j^ε; note that $m(0) = \gamma_j$. Then Sender's payoff as a function of ε on B_j is

$$P(\varepsilon) = \left[\frac{m(\varepsilon) - \gamma_j}{\gamma_{j+1} - \gamma_j} v_{j+1} + \frac{\gamma_{j+1} - m(\varepsilon)}{\gamma_{j+1} - \gamma_j} v_j \right] [F(\bar{b}_j) - F(\underline{b}_j + \varepsilon)] + v_{j-1}[F(\underline{b}_j + \varepsilon) - F(\underline{b}_j)].$$

To show that this is a profitable deviation, it suffices to show that $P'(0) := \lim_{\varepsilon \searrow 0} P'(\varepsilon) > 0$. Algebra reveals that

$$P'(\varepsilon) = f(\underline{b}_j + \varepsilon) \left[v_{j-1} - \frac{\gamma_{j+1} v_j - \gamma_j v_{j+1}}{\gamma_{j+1} - \gamma_j} - (\underline{b}_j - \varepsilon) \frac{v_{j+1} - v_j}{\gamma_{j+1} - \gamma_j} \right];$$

consequently, as $\varepsilon \searrow 0$,

$$P'(0) = \frac{f(\underline{b}_j)}{\gamma_{j+1} - \gamma_j} \left[(\gamma_j - \underline{b}_j)(v_{j+1} - v_j) - (\gamma_{j+1} - \gamma_j)(v_j - v_{j-1}) \right]. \tag{8}$$

Because $f(\underline{b}_j) > 0$ and $\gamma_{j+1} - \gamma_j > 0$ by assumption, the sign of $P'(0)$ is the same as the sign of term in the square brackets in the right-hand side of (8). Thus, $P'(0) > 0$

[42] If $i > j + 1$, it must be that $\bar{b}_j > \gamma_i > \gamma_{j+1}$, and hence one can work with a_{j+1} instead.

if and only if

$$\frac{v_{j+1} - v_j}{\gamma_{j+1} - \gamma_j} > \frac{v_j - v_{j-1}}{\gamma_j - \underline{b}_j}.$$

Because $h(\gamma_j; \gamma_{j+1}) > \underline{b}_j$ by definition, (2) implies the inequality above.

Case 2. B_j is a component of a nested intervals representation, there are three cases: (a) $\ell < j < i$, (b) $j < \ell < i$, and (c) $j < i < \ell$. In Cases (b) and (c), B_j must be the middle interval, and hence they are completely analogous to Case 1; therefore, it remains to consider Case (a). In this case, $B_\ell = [\underline{b}_\ell, \overline{b}_\ell]$, and $B_j = [\underline{b}_j, \underline{b}_\ell] \cup [\overline{b}_\ell, \overline{b}_j]$. WLOG, assume that $\underline{b}_j < \underline{b}_\ell$; by Lemma A.6, $\{\gamma_\ell, \gamma_j\}$ is feasible for $[\underline{b}_j, \overline{b}_j]$, and

$$\mathbb{E}[\omega \mid \omega \in [y(\gamma_\ell), \overline{b}_j]] > \gamma_j. \tag{9}$$

Furthermore, it must be that $\gamma_i \in [\overline{b}_\ell, \overline{b}_j]$: if instead $\gamma_i \in [\underline{b}_j, \underline{b}_\ell]$, it must be that $\gamma_i < \gamma_\ell$, which violates the assumption that $\ell < i$.

For $z \in (\gamma_i, \overline{b}_j)$, let $h(z)$ solve $\mathbb{E}[\omega \mid \omega \in [\underline{b}_j, h(z)] \cup [z, \overline{b}_j]] = \gamma_i$. By (9) and Lemma A.6, for z close enough to $\overline{b}_j$, $\{\gamma_\ell, \gamma_j\}$ is feasible for $[h(z), z]$. Consequently, one can find a profitable deviation in a similar manner as in Case 1 (I) *mutatis mutandis*.

Now suppose condition (ii) is violated; that is, for any nested intervals representation $\{B_L, B_H\}$, $\gamma_H < \overline{b}_L$. Since $B_L = [\underline{b}_L, \overline{b}_L]$, this case is isomorphic to Case 1 when condition (i) is violated. As a consequence, (2) implies that (ii) must also hold for all bi-pooling solutions.

Therefore, every bi-pooling solution must satisfy both (i) and (ii), and hence implementable. This completes the proof. ∎

A.1.6 Proof of Corollary 3.8

By definition of $h(\gamma_i; \gamma_{i+1})$, when f is increasing, $\gamma_{i+1} - \gamma_i \leq \gamma_i - h(\gamma_i; \gamma_{i+1})$. Then since $\gamma_{i+1} - \gamma_i \leq \gamma_i - \gamma_{i-1}$, it must be that $\gamma_{i+1} - \gamma_i \leq \min\{\gamma_i - \gamma_{i-1}, \gamma_i - h(\gamma_i; \gamma_{i-1})\}$. Now there are two cases. If $v_{i+1} - v_i > v_i - v_{i-1}$, (2) must hold; then by Proposition 3.7, every bi-pooling solution can be implemented. If instead $\gamma_{i+1} - \gamma_i > \gamma_i - \gamma_{i-1}$, it must be that $\gamma_{i+1} - \gamma_i < \min\{\gamma_i - \gamma_{i-1}, \gamma_i - h(\gamma_i; \gamma_{i-1})\}$; this inequality and $v_{i+1} - v_i \geq v_i - v_{i-1}$ together imply (2). Again by Proposition 3.7, every bi-pooling solution can be implemented.

A.2 Proofs for Section 4

A.2.1 Proof of Lemma 4.1

Suppose that a deterministic representation $\{B_i\}_{i=0}^{n-1}$ is both obedient and incentive compatible. Consider the tuple (σ, τ, p) defined in the proof of Lemma A.2. By obedience, when Receiver sees message B_i, it is a best response to play a_i with probability 1; and by incentive compatibility, in any state Sender does not want to deviate to full disclosure. Finally, the choice of the belief system guarantees that any other deviations are not profitable either. Consequently, (σ, τ, p) is an equilibrium, and hence the deterministic representation $\{B_i\}_{i=0}^{n-1}$ defines an ORE by construction.

Suppose that a deterministic representation $\{B_i\}_{i=0}^{n-1}$ defines an ORE. Obedience must hold because $\tau(a_i \,|\, B_i) = 1$ for all i implies that $\mathbb{E}[\omega \,|\, \omega \in B_i] \in [\gamma_i, \gamma_{i+1}]$ for all i. Incentive compatibility follows since for every $\omega \in [0, 1]$ there exists $i \in \{0, 1, \ldots, n-1\}$ such that $\sigma(B_i \,|\, \omega) = 1$ is a best response, which implies that deviating to $m = \{\omega\}$ is not a profitable deviation. Thus, $\{B_i\}_{i=0}^{n-1}$ is both obedient and incentive compatible.

A.2.2 Proof of Proposition 4.2

I start by presenting a roadmap for the proof:

1. Lemma A.7 discovers that the search for Sender's preferred equilibrium can be confined to the set of OREs.
2. Lemma A.8 establishes (i) b. of the proposition.
3. Lemma A.9, Lemma A.10, and Lemma A.11 together indicate that one can further restrict attention to OREs defined by laminar representations.
4. Lemma A.12 reveals that every element of a laminar representation defining a Sender's preferred ORE is the union of at most $n-1$ disjoint intervals.
5. Lemma A.13 shows that a Sender's preferred equilibrium exists.
6. Lemma A.14 argues that Sender's ex ante payoff in her preferred equilibrium is strictly higher than in any equilibrium that features unraveling.

Lemma A.7. *If there exists a Sender's preferred equilibrium* (σ, τ, p)*, then there exists an ORE that generates the same ex ante payoff to Sender.*

Proof of Lemma A.7. Because (σ, τ, p) is a Sender's preferred equilibrium, Receiver must break ties in favor of Sender; consequently, $\tau(a_i \,|\, m) = 1$ if and only if $\mathbb{E}[\omega \,|\, \omega \in m] \in [\gamma_i, \gamma_{i+1})$. Suppose in some $\omega \in [0, 1]$, $|\operatorname{supp}\, \sigma(\cdot \,|\, \omega)| > 1$. Because (σ, τ, p) is an equilibrium, for every $m, m' \in \text{supp } \sigma(\cdot \mid \omega)$, there must exist $i \in \{0, 1, \ldots, n-1\}$ such that $\tau(a_i \mid m) = \tau(a_i \mid m') = 1$. As a consequence, for every $\omega \in [0, 1]$, there is an action a_i played with probability 1. For $i = 0, 1, \ldots, n-1$, let

$$B_i = \text{cl}\{\omega \in [0,1] : \tau(a_i \mid m) = 1 \text{ for all } m \in \text{supp } \sigma(\cdot \mid \omega)\},$$

and it is straightforward that $\{B_i\}_{i=0}^{n-1}$ is a deterministic representation. I claim that it is both obedient and incentive compatible. To see obedience, note that for all $\omega \in B_i$, for all $m \in \text{supp}\, \sigma(\cdot \mid \omega)$, $\mathbb{E}[\omega \mid \omega \in m] \in [\gamma_i, \gamma_{i+1}]$, and also $\omega \in m$; therefore, $\mathbb{E}[\omega \mid \omega \in \cup_{\omega \in B_i, m \in \text{supp}\, \sigma(\cdot \mid \omega)} m] \in [\gamma_i, \gamma_{i+1}]$, and $B_i \subseteq \cup_{\omega \in B_i, m \in \text{supp}\, \sigma(\cdot \mid \omega)} m$. Thus, it must be that $\mathbb{E}[\omega \mid \omega \in B_i] \in [\gamma_i, \gamma_{i+1}]$. Moreover, incentive compatibility follows because (σ, τ, p) is an equilibrium, and hence in any $\omega \in B_i$ using $m = \{\omega\}$ cannot increase Sender's payoff. Then by Lemma 4.1, there exists an ORE defined by the deterministic representation $\{B_i\}_{i=0}^{n-1}$. By construction, this ORE generates the same ex ante payoff to Sender. ■

Consequently, it suffices to focus on Sender's preferred ORE henceforth. Lemma A.8 provides a useful characterization of such OREs.

Lemma A.8. *Let a Sender's preferred ORE be defined by deterministic representation $\{B_i\}_{i=0}^{n-1}$, then it must be that $\mathbb{E}[\omega \mid \omega \in B_j] \in [\gamma_j, \gamma_{j+1})$ and $\mathbb{E}[\omega \mid \omega \in B_k] = \gamma_k$ for all k with $k > j$ such that a_k is not skipped.*

Proof of Lemma A.8. Suppose to the contrary that $\{B_i\}_{i=0}^{n-1}$ has $\mathbb{E}[\omega \mid \omega \in B_k] > \gamma_k$ for some $k > j$ where a_k is not skipped. Because $\{B_i\}_{i=0}^{n-1}$ defines an ORE, it must be both obedient and incentive compatible. Let $\underline{b}_j = \inf\left(\text{int}\,(B_j)\right)$; because $f > 0$, there exists $x \in B_j$ with $x > \underline{b}_j$ such that $\mathbb{E}[\omega \mid \omega \in [\underline{b}_j, x] \cup B_k] \geq \gamma_k$, and clearly $\mathbb{E}[\omega \mid \omega \in B_j \backslash [\underline{b}_j, x]] > \gamma_j$. Now consider the deterministic representation $\{\widetilde{B}_i\}_{i=0}^{n-1}$ where $\widetilde{B}_j = B_j \backslash [\underline{b}_j, x]$, $\widetilde{B}_k = [\underline{b}_j, x] \cup B_k$, and $\widetilde{B}_i = B_i$ for $i \neq j, k$. It can be seen that the new deterministic representation strictly improves Sender's expected payoff, and it is still obedient and incentive compatible. A contradiction. ■

Because an ORE can be identified by the deterministic representation associated with it, a Sender's preferred ORE induces a distribution over posterior mean G_S with support on at most n points, which must be a MPC of the prior F.[43] In particular, by Lemma A.8, it must be that $\text{supp}(G_S) \subseteq \{\gamma_i\}_{i=1}^{n-1} \cup \{\gamma_S\}$, where $\gamma_S \in [0, 1] \backslash \{\gamma_i\}_{i=1}^{n-1}$.

[43] See Footnote 6 for the definition of a mean-preserving contraction (MPC).

Before proceeding to further characterize the structure of Sender's preferred OREs, I introduce the following auxiliary result first.

Lemma A.9. *There exist* $0 = d_0 \leq d_1 \leq \dots \leq d_{m-1} \leq d_m = 1$ *with* $m \leq n$ *such that* $\int_0^x G_S(q)\,\mathrm{d}q \leq \int_0^x F(q)\,\mathrm{d}q$ *on* $[d_i, d_{i+1}]$ *for all* $i = 0, 1, \dots, m-1$, *and the inequality holds with equality only at* d_i *and* d_{i+1}.

The proof of Lemma A.9 is relegated to the Supplementary Appendix. Intuitively, because the prior F is strictly increasing, and G_S is a step function with at most n jumps, the state space can be partitioned into at most n intervals on the interior of which the MPC constraint does not bind.

Lemma A.10 shows that G_S induces a laminar representation. As it mostly follows from the results in Candogan and Strack (2022), its proof is relegated to the Supplementary Appendix.

Lemma A.10. *If* G_S *is the posterior mean distribution that is induced by a Sender's optimal ORE, there exists a laminar representation that is induced by it.*

Lemma A.11 shows that in searching for Sender's preferred equilibrium, one can further restrict attention to OREs defined by laminar representations.

Lemma A.11. *For every deterministic representation that is associated with a Sender's preferred ORE, there exists a laminar representation that does the same.*

Proof of Lemma A.11. By Lemma 4.1, it suffices to show that there exists a laminar representation that is both obedient and incentive compatible, and generates the same ex ante payoff to Sender. Fix a deterministic representation $\{B_i\}_{i=0}^{n-1}$ associated with a Sender's preferred ORE, and denote the posterior mean distribution that corresponds to it by G^*. By Lemma A.8, $\text{supp}(G^*) \subseteq \{\gamma_i\}_{i=1}^{n-1} \cup \{\gamma_S\}$; and by Lemma A.10, G^* induces a laminar representation $\{B_i^L\}_{i=0}^{n-1}$. Because the two representations are induced by the same posterior mean distribution, it must be that that (i) $\text{int}(B_i) = \emptyset$ if and only if $\text{int}(B_i^L) = \emptyset$, and (ii) for every k such that $\text{int}(B_k) \neq \emptyset$, $\mathbb{E}[\omega \,|\, \omega \in B_k] = \mathbb{E}[\omega \,|\, \omega \in B_k^L] \in \{\gamma_k, \gamma_S\}$ and $\mu_F(B_k) = \mu_F(B_k^L) \in \{g^*(\gamma_k), g^*(\gamma_S)\}$, where g^* is the pmf of G^*. Thus, the two representations generate the same ex ante payoff to Sender, and $\{B_i^L\}_{i=0}^{n-1}$ must be obedient because $\{B_i\}_{i=0}^{n-1}$ is. Consequently, it only suffices to show that $\{B_i^L\}_{i=0}^{n-1}$ satisfies (IC).

To this end, consider any $m = 1, \dots, n-1$; I need to show that $A_m \subseteq \cup_{i \geq m} B_i$. Suppose first that a_m is not skipped. By Lemma A.9, there must exist an interval I on which the MPC constraint only binds at the endpoints, and $\gamma_m \in I$. Moreover, for every $\gamma_k \in (\text{supp}\,(G^*) \cap I)$, it must be that $B_k \subseteq I$ and $B_k^L \subseteq I$.

If $\gamma_m = \min(\text{supp}\,(G^*) \cap I)$, it must be that $\cup_{i\geq m} B_i^L = \cup_{i\geq m} B_i$, and thus $A_m \subsetneq \cup_{i\geq m} B_i$ implies $A_m \subsetneq \cup_{i\geq m} B_i^L$. Consider next that $\gamma_m > \min(\text{supp}\,(G^*) \cap I)$. Clearly,

$$\gamma_m \geq \sup(\text{int}\,(B_k)) \text{ for all } k < m \text{ such that } \text{int}\,(B_k) \neq \emptyset, \tag{10}$$

as otherwise (IC) is violated.[44] Because $\{B_i^L\}_{i=0}^{n-1}$ is laminar, either $\text{co}\,(B_{m-1}^L) \cap \text{co}\,(B_m^L) = \emptyset$ or $\text{co}\,(B_{m-1}^L) \subseteq \text{co}\,(B_m^L)$.[45] In the former case, B_m^L must be an interval; consequently, $\gamma_m \in B_m^L$. Furthermore, the laminar structure of $\{B_i^L\}_{i=0}^{n-1}$ implies that for any $\omega > \sup B_m^L$, $\omega \in B_q^L$ for some $q > m$, and hence $A_m \subsetneq \cup_{i\geq m} B_i^L$. In the latter case, to show that $A_m \subsetneq \cup_{i\geq m} B_i^L$, it suffices to show that $\gamma_m \geq \sup(\text{co}\,(B_{m-1}^L))$.[46] The laminar structure implies that $\text{co}\,(B_{m-1}^L)$ is an interval. Let

$$\mathcal{N} = \left\{ i \leq m-1 \,:\, \gamma_i \in \text{supp}\,(G^*) \cap I, \text{ and } \text{co}\,(B_i^L) \subseteq \text{co}\left(B_{m-1}^L\right) \right\},$$

and define $J = \cup_{i\in\mathcal{N}} B_i$. By (10), $\sup(\text{int}\,(J)) \leq \gamma_m$; thus, to show that $\gamma_m \geq \sup(\text{co}\,(B_{m-1}^L))$, I only need to prove that $\sup(\text{int}\,(J)) \geq \sup(\text{co}\,(B_{m-1}^L))$. Suppose not, so $\sup(\text{int}\,(J)) < \sup(\text{co}\,(B_{m-1}^L))$. Because $f > 0$, $\mu_F\,(\text{int}\,(J)) = \mu_F\left(\text{co}\,(B_{m-1}^L)\right)$, and that $\text{co}\,(B_{m-1}^L)$ is an interval, it must be that $\mathbb{E}[\omega \,|\, \omega \in \text{int}(J)] < \mathbb{E}[\omega \,|\, \omega \in \text{co}\,(B_{m-1}^L)]$. However, $\mathbb{E}[\omega \,|\, \omega \in \text{int}\,(J)] = \mathbb{E}[\omega \,|\, \omega \in \text{co}\,(B_{m-1}^L)]$ by construction; a contradiction. Therefore, $\gamma_m \geq \sup(\text{co}\,(B_{m-1}^L))$, and hence $A_m \subsetneq \cup_{i\geq m} B_i^L$.

Finally, consider the case that a_m is skipped. If $m < \min\{i \,:\, \text{int}\,(B_j) \neq \emptyset\}$, we are done. Otherwise, define $a_k = \max\{a_i \in A \,:\, a_i \text{ is not skipped, and } i < m\}$, and $a_q = \min\{a_i \in A \,:\, a_i \text{ is not skipped, and } i > m\}$. Replace m and $m-1$ by q and k, respectively, in the previous paragraph, it can be shown that $A_m \subsetneq \cup_{i\geq q} B_i^L = \cup_{i\geq m} B_i^L$. This completes the proof. ■

Lemma A.12 establish an important property of laminar representations associated with Sender's preferred OREs.

[44] This is obvious for k's with $k < m$ and $\mathbb{E}[\omega \,|\, \omega \in B_k] \notin \text{supp}\,(G^*) \cap I$. If $\gamma_k \in \text{supp}\,(G^*) \cap I$, the only possibility that makes $\gamma_m \leq \sup(\text{int}\,(B_k))$ hold without violating $A_m \subsetneq \cup_{i\geq m} B_i$ is that $A_m \subsetneq \text{co}\,(\text{int}\,(B_k))$. Then there must exist $q > m$ such that $A_q \subsetneq \cup_{i\geq q} B_i$, a contradiction.

[45] Here I am assuming that a_{m-1} is not skipped. This is just for simplifying notation: if instead a_{m-1} is skipped, replace it with the action immediately below a_m in the set of unskipped actions.

[46] This is because, again, the laminar structure guarantees that (1) $B_m \subsetneq \text{co}\,(B_m) \backslash \left(\cup_{i\in I, i<m} \text{co}\,(B_i)\right)$, and (2) for any $\omega > \sup(\text{co}\,(B_m))$, $\omega \in B_q$ for some $q > m$, and hence $A_m \subsetneq \cup_{i\geq m} B_i^L$.

Lemma A.12. *Let $\{B_i\}_{i=0}^{n-1}$ be a laminar representation associated with a a Sender's preferred ORE. Every member of $\{B_i\}_{i=0}^{n-1}$ is the union of at most $n-1$ intervals.*

Proof of Lemma A.12. Recall that a laminar representation is defined by (5). Because $\text{co}\,(B_k)$ must be an interval for all k such that $\text{int}\,(B_k) \neq \emptyset$, $\cup_{k<i}\,\text{co}\,(B_k)$ is the union of at most i intervals. By Claim C.1 in the Supplementary Appendix, since $\{B_i\}_{i=0}^{n-1}$ defines a Sender's preferred ORE, $\text{int}\,(B_0) \cap B_k = \emptyset$ for all $k \geq 1$. Hence, if B_0 has nonempty interior, it must be an interval. Thus, by taking out $\cup_{k<i}\,\text{co}\,(B_k)$ from $\text{co}\,(B_i)$, at most $i-1$ intervals are removed, and hence the remainder, namely $\text{co}\,(B_i) \setminus \cup_{k<i}\,\text{co}\,(B_k)$, must be the union of at most i intervals. By taking closure, B_i is also the union of at most i intervals. Because $i \leq n-1$, the lemma thus follows. ∎

Lemma A.11 and Lemma A.12 together imply that, in a Sender's preferred ORE, the associated deterministic representation is laminar, and each of the elements of the laminar representation is the union of at most $n-1$ intervals. These properties of Sender's preferred ORE allow me to prove the existence of Sender's preferred equilibrium.

Lemma A.13. *A Sender's preferred equilibrium exists.*

Since the proof of Lemma A.13 is mostly technical, it is deferred to the Supplementary Appendix. Lastly,

Lemma A.14. *A Sender's preferred ORE must generate a strictly higher ex ante payoff to Sender than the unraveling payoff.*

Proof of Lemma A.14. Observe that Sender's ex ante payoff in an unraveling equilibrium is the same as the ORE defined by the deterministic representation $\{A_i\}_{i=0}^{n-1}$. Let a Sender's preferred ORE be defined by $\{B_i\}_{i=0}^{n-1}$; it must generate a weakly higher payoff to Sender than $\{A_0, A_1, \ldots, A_{n-1}\}$ because incentive compatibility guarantees that in every $\omega \in B_i \setminus \cup_{j>i} B_j$ and all $i = 0, 1, \ldots, n-1$, sending message B_i makes Sender weakly better off than truthfully revealing the state as in the unraveling equilibrium. Moreover, for incentive compatibility to hold, action a_{n-1} cannot be skipped; consequently, $\mu_F(B_{n-1}) > 0$. Because $\mathbb{E}[\omega \,|\, \omega \in B_{n-1}] = \gamma_{n-1} = \inf A_{n-1}$, $f > 0$ implies that $\mu_F(B_{n-1} \setminus A_{n-1}) > 0$. Consequently, in a positive measure of states, Sender is strictly better off in Sender's preferred ORE than in an unraveling equilibrium, which implies that her ex ante payoff must be strictly higher in the former. ∎

A.2.3 Proof of Proposition 4.5

By Proposition 4.2, one can find a laminar representation that corresponds to Sender's preferred equilibrium, denote it by $\{B_i^S\}_{i=0}^{n-1}$; and let Sender's ex ante payoff in this equilibrium be R^S. Clearly, $\{A_i\}_{i=0}^{n-1}$ is a laminar representation that corresponds to an ORE and yields Sender the same ex ante payoff as in the unraveling equilibrium; let Sender's ex ante payoff in this equilibrium be R^U. The "only if" direction is straightforward. By definition, in no equilibrium can Sender achieve an ex ante payoff strictly higher than R_S. Moreover, in an ORE defined by a deterministic representation $\{B_i\}_{i=0}^{n-1}$, Sender's ex ante payoff is given by $\sum_{i=0}^{n-1} v(a_i)\mu_F(B_i)$; then (IC) implies that $R = \sum_{i=0}^{n-1} v(a_i)\mu_F(B_i) \geq \sum_{i=0}^{n-1} v(a_i)\mu_F(A_i) = R_U$.

The "if" direction requires more work. By Lemma 4.1, it suffices to show that for any $R \in [R^U, R^S]$, one can find an obedient and incentive compatible laminar representation $\{B_i\}_{i=0}^{n-1}$ that generates R. To this end, for all $z \in [0, \gamma_1]$, let $\hat{B}_0^z = B_0^S \cup [0, z]$ and $\hat{B}_i^z = B_i^S \backslash [0, z]$ for all $i \geq 1$. Then for any $z \in [0, \gamma_1]$, because $\mathbb{E}\left[\omega \middle| \omega \in \hat{B}_0^z\right] \in A_0$, $\mathbb{E}\left[\omega \middle| \omega \in \hat{B}_0^z\right] \in A_0$ by construction; and for all $i \geq 1$, $\mathbb{E}\left[\omega \middle| \omega \in \hat{B}_i^z\right] \in A_i$ since $\mathbb{E}\left[\omega \middle| \omega \in \hat{B}_i^S\right] \in A_i$. Consequently, $\{\hat{B}_i^z\}_{i=0}^{n-1}$ is obedient. To see that it is also incentive compatible, observe that

$$\bigcup_{j \geq i}^{n-1} \hat{B}_j^z = \left(\bigcup_{j \geq i}^{n-1} B_j^S\right) \backslash [0, z] \text{ for all } i \geq 1,$$

and hence if $A_j \subseteq \cup_{j \geq i}^{n-1} B_j^S$ it must be that $A_j \subseteq \cup_{j \geq i}^{n-1} \hat{B}_j^z$. Then by Lemma 4.1, for every $z \in [0, \gamma_1]$ there exists an equilibrium that corresponds to $\{\hat{B}_i^z\}_{i=0}^{n-1}$.

Let $P(z)$ denote Sender's ex ante payoff in the equilibrium that corresponds to $\{\hat{B}_i^z\}_{i=0}^{n-1}$: $P(z) = \sum_{i=0}^{n-1} v(a_i)\, \mu_F\left(\hat{B}_0^z\right)$; because $f > 0$, $P(z)$ is continuous in z. Since $P(0) = R^S > R$, if $P(\gamma_1) := R_1 \leq R$, by the intermediate value theorem, there exists $z^* \in (0, \gamma_1]$ such that $P(z^*) = R$. If instead $R_1 > R$, I start from $\{\hat{B}_i^{\gamma_1}\}_{i=0}^{n-1}$, and for every $z \in [\gamma_1, \gamma_2]$, define $\hat{B}_0^z = \hat{B}_0^{\gamma_1}$, $\hat{B}_1^z = \hat{B}_1^{\gamma_1} \cup [\gamma_1, z]$, and $\hat{B}_i^z = B_i^s \backslash [v_1, z]$ for all $i \geq 2$. It is routine to check that $\{\hat{B}_i^z\}_{i=0}^{n-1}$ is obedient and incentive compatible. Because $P(\gamma_1) = R_1 > R$, if $P(\gamma_2) : R_2 \leq R$, by the intermediate value theorem, there exists $z^* \in (\gamma_1, \gamma_2]$ such that $P(z^*) = R$. If instead $R_2 < R$, proceed in a kindred manner *mutatis mutandis*. Because $P(\gamma_{n-1}) = R^U < R$, eventually one must be able to find some $z^* \in [0, \gamma_{n-1}]$ such that $P(z^*) = R$. Consequently, by letting $B_i = \hat{B}_i^z$ for all i, $\{B_i\}_{i=0}^{n-1}$ identifies an equilibrium with Sender's ex ante payoff being R. It can be checked that this is a laminar representation. This completes the proof.