ECN594: Math Bootcamp

Practice Questions III

August 2022

1. Prove Proposition 2.
2. For each of the following functions, determine

i. whether the extreme value theorem implies that the function has a maximum and a minimum, and

ii. if the extreme value theorem does not apply, whether the function does in fact have a maximum and/or a minimum.

(a) $f(x) = x^2$ on the interval $[-1, 1]$;

(b) $f(x) = x^2$ on the interval $(-1, 1)$;

(c) $f(x) = |x|$ on the interval $[-1, \infty)$;

(d) $f(x)$ defined by $f(x) = 1$ if $x < 0$ and $f(x) = x$ if $x \geq 0$ on the interval $[-1, 1]$;

(e) $f(x)$ defined by $f(x) = 1$ if $x < 0$ and $f(x) = x$ if $x \geq 0$, on the interval $(-\infty, \infty)$;

(f) $f(x)$ defined by $f(x) = x^2$ if $x < 0$ and $f(x) = x$ if $x \geq 0$ on the interval $[-1, 1]$.

3. Consider $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ defined by

$$f(x) = \begin{cases} -x & \text{if } x < 0 \\ 0 & \text{if } x \geq 0 \end{cases}$$

on the interval $D = [-1, \infty)$.

(a) Which assumptions of the extreme value theorem hold and which fail for this problem? Explain.

(b) Does f in fact attain a maximum on D? If yes, write down all of its maximizers on D; if not, explain why. Does the Weierstrass extreme value theorem provide necessary or sufficient conditions, or both, of the existence of a maximum?

4. Prove Rolle's theorem:

Theorem (Rolle)**.** *Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous, and differentiable on (a, b). If $f(a) = f(b) = 0$, then there exists $\theta \in (a, b)$ such that $f'(\theta) = 0$.*

Hint. (1) Observe that f is continuous on $[a, b]$; (2) Use the first-order condition.

5. A consumer has the utility function

$$u(x_1, x_2) = x_1 x_2,$$

has income $y > 0$ and faces the prices $p_1 > 0$ and $p_2 > 0$. She is required to spend all her income.

(a) Formulate her utility maximization problem.

(b) Transform this problem into a maximization problem in the single variable x_1.

Hint. Isolate x_2 in the budget constraint and substituting it into the utility function.

(c) Find the consumption bundle (x_1, x_2) that maximizes the consumer's utility.

6. Let $f : [0, 1] \longrightarrow \mathbb{R}$ be a *strictly convex* function.

(a) Prove that if x^* maximizes f on $[0, 1]$, then either $x^* = 0$ or $x^* = 1$.

(b) Does it follow that f has a maximizer on $[0, 1]$? If yes, prove it; if no, give a counterexample (a carefully-explained picture will suffice if a counterexample is needed).

Hint. You might want to use the following fact: let $S \subseteq \mathbb{R}$ and let $f : S \longrightarrow \mathbb{R}$ be either convex or concave, then f is continuous on the interior of S. See, for example, Sundaram (1996), page 177, Theorem 7.3 for a proof.

7. The equilibrium value of the variable x is the solution of the equation

$$f(x, \alpha, \beta) + g(u(x), v(\alpha)) = 0,$$

where α and β are parameters and f, g, u, and v are continuously differentiable functions. Use the implicit function theorem to find that, holding β constant, how is the equilibrium value of x affected by a change in the parameter α?

8. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a differentiable function. Consider the following optimization problem: Choose x to maximize $f(x)$ subject to i) a nonnegativity constraint that $x \geq 0$ and; ii) a constraint that $x \leq 1$. The problem can be written compactly as

$$\max_{x \in [0,1]} f(x).$$

(a) Write down the KKT conditions for this problem.

(b) Suppose that x^* is the *unique* point in $\mathbb{R}$ that solves the KKT conditions; that is, (x^*, λ^*) solves the KKT conditions and if (x', λ') solves the KKT conditions, then $x^* = x'$. Show that x^* indeed solves the optimization problem.

9. Show that the saddlepoint condition (Corollary 3) is sufficient.

Hint. Use the previous corollary (Corollary 2).

10. Solve the problem

$$\begin{aligned}
\max_{(x_1,x_2)\in\mathbb{R}^2} \quad & -x_1^2 - x_1x_2 - x_2^2 \\
\text{s.t.} \quad & -x_1 + 2x_2 \geq 1 \\
& 2x_1 + x_2 \leq 2.
\end{aligned}$$

Note. You may use without proof the fact that

$$f(x_1, x_2) = -x_1^2 - x_1x_2 - x_2^2$$

is concave.

11. A firm produces a single good that it sells at a unit price of 10. The cost of production is $c(q) = \frac{1}{2}q^2$. The firm's objective is to choose output q to maximize revenue $10q$, subject to the constraint that $q \geq 0$ and profit is at least 50: $10q - \frac{1}{2}q^2 \geq 50$.

(a) Which ones of the constraint qualifications ((A),(B) and (C)) apply and which ones do not? Explain.

(b) Write down the KKT conditions for the firm's problem.

(c) Identify the points that satisfy the KKT conditions.

(d) Find the solutions of this problem.

12. Alejandro consumes two goods, food (y) and maté (x). His utility function is

$$u(x, y) = x^2y^2.$$

The price of maté is 2 and the price of food is 1, and his income is 2. So his utility maximization problem is

$$\begin{aligned}
\max_{(x,y)\in\mathbb{R}^2} \quad & u(x, y) \\
\text{subject to} \quad & 2x + y \leq 2 \\
& x \geq 0 \\
& y \geq 0.
\end{aligned}$$

(a) Argue that this problem has a solution.
Hint. Use the extreme value theorem; to check one of the assumptions of that theorem, a carefully drawn picture will suffice.

(b) Are the Karush-Kuhn-Tucker (KKT) conditions necessary for a solution to this problem?

(c) Write down the KKT conditions for this problem.

(e) Solve the KKT conditions and find the solutions of the problem. Argue that why do the solutions you identify indeed solve this problem.

13. Find the values of (x_1, x_2) that maximize

$$u(x_1, x_2) = (1/a)x_1^a + x_2,$$

subject to the constraints that

$$\begin{aligned} p_1x_1 + p_2x_2 &\leq w, \\ x_1 &\geq 0, \\ x_2 &\geq 0, \end{aligned}$$

where $a > 0$, $p_i > 0$ for $i = 1, 2$, and $w > 0$.

Hint. There are two (broad) cases: $a < 1$ and $a \geq 1$.

References

Sundaram, R. K. (1996): *A First Course in Optimization Theory*, New York: Cambridge University Press.