ECN594: Math Bootcamp

Practice Questions II

August 2022

1. Let (x^n) be a real sequence. Show that if $|x^n| \longrightarrow 0$, then $x^n \longrightarrow 0$.

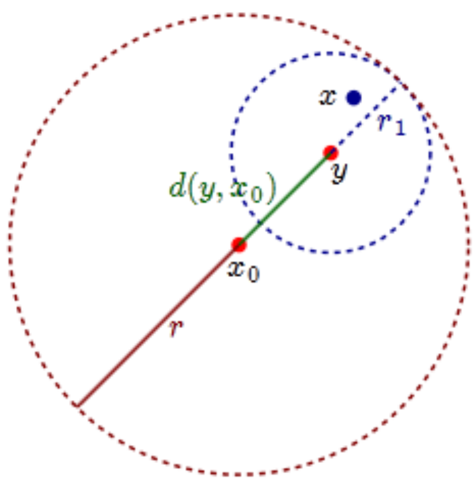
Hint: Use Proposition 1 (the squeeze theorem).

2. Prove Claim 2: Every convergent real sequence is bounded.

Hint: Fix $\varepsilon = 1$.

3. Let (a^n) and (b^n) be real sequences, and suppose $a^n \longrightarrow a$, and $b^n \longrightarrow b$. Show that if $a^n \leq b^n$ for all $n \in \mathbb{N}$, then $a \leq b$.

4. Prove Claim 3: An open ball is an open set.



Hint: Fix an arbitrary open ball $B(x_0, r)$. For any $y \in B(x_0, r)$, follow the picture above to choose some $r_1 > 0$ such that

$$B(y, r_1) \subseteq B(x_0, r).$$

5. Let (x^k) be a sequence, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be a function. Negate the following statements:

(a) for all $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ such that $d(x^k, x) < \varepsilon$ for all $k \geq N$;

(b) for every $Q \in \mathbb{R}$, there exists $M \in \mathbb{N}$ with $x^k \geq Q$ for each $k \geq M$;

(c) for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in \mathbb{R}^n$ with $|x - y| < \delta$, we have $|f(x) - f(y)| < \varepsilon$.

6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be continuous. Show that the composite function $h = f \circ g$ is continuous.

7. Let D be a nonempty, finite set. Show that f is continuous at each point in its domain.

8. Show that the function $g : \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by

$$g(x, y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$$

is not continuous at (0,0).

9. Show that $\cos x - \cos y \leq |x - y|$ for any $x, y \in \mathbb{R}$.

Hint. $(\cos x)' = -\sin x$, and $\sin x \in [-1, 1]$ for all $x \in \mathbb{R}$. Now use Theorem 7 (the mean value theorem).

10. Prove Theorem 8.

Hint. Use Theorem 7 (the mean value theorem).

11. Let $f : \mathbb{R}\backslash\{-1\} \longrightarrow \mathbb{R}$ be defined as

$$f(x) = \frac{x-1}{x+1}.$$

Let g denote the inverse of f, which can be written as

$$g(y) = \frac{1+y}{1-y}.$$

Use the inverse function theorem and direct computation to get the derivative of g.

12. Prove Proposition 8.

13. Prove the *Young's inequality*: For all $p, q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$,

$$a^{\frac{1}{p}} b^{\frac{1}{q}} \leq \frac{a}{p} + \frac{b}{q}$$

for any $a, b \in \mathbb{R}$.

Hint. Use the facts that $f(x) = \log x$ is strictly increasing and strictly concave on $(0, \infty)$, and

$$\log y^m z^k = m \log y + k \log z.$$