ECN594: Math Bootcamp

# Optimization

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Many parts of these notes are largely based on Chapter 2-6 of Sundaram (1996), and especially Chapter 4-7 of Osborne (2016). I have also benefited from Chapter 5 of Vohra (2005), Chapter 15 and 18 of Simon and Blume (1994), Chapter 7 of de la Fuente (2000), and lecture notes written by (in random order) Andreas Kleiner, Ahmet Altınok, and Hao-Che (Howard) Hsu.

# Contents

# 1 Optimization problems: An introduction

In these notes, we focus on optimization problems that the values of a given function $f : S \longrightarrow \mathbb{R}$ are to be maximized or minimized over a set $D$, where $D \subseteq S \subseteq \mathbb{R}^n$. We call the function $f$ as the **objective function**, and the set $D$ is called the **constraint set**. We can describe a maximization problem as follows:

$$\text{maximize } f(x) \quad \text{subject to } x \in D;$$

or more compactly,

$$\max_{x \in D} f(x).$$

A minimization problem can be described analogously.

A **solution** to a maximization problem is a point $x \in D$ such that $f(x) \geq f(y)$ for all $y \in D$. We say that $f$ attains a **maximum** on $D$ at $x$, and also refer to $x$ as a **maximizer** of $f$ on $D$; we usually write the set of all maximizers of this problem as

$$\arg\max_{x \in D} f(x).$$

Analogously, $x$ is a solution for a minimization problem if $f(x) \leq f(y)$ for all $y \in D$. And we say $f$ attains a **minimum** on $D$ at $x$, so $x$ is a **minimizer** of $f$ on $D$; we usually write the set of all minimizers of this problem as

$$\arg\min_{x \in D} f(x).$$

Notice that, as the following two example point out, it is not guaranteed that a solution exists or it is unique.

**Example 1.** Let $D = \mathbb{R}_+$ and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^2$. Consider the maximization problem $\max_{x \in D} f(x)$: we have $f(D) = \mathbb{R}_+$, and $\sup f(D) = \infty$, so this maximization problem has no solution.

**Example 2.** Let $E = [0, 1)$ and $g : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $g(x) = x$. Consider the maximization problem $\max_{x \in E} g(x)$, we have $f(E) = [0, 1)$, and $\sup f(E) = 1$. However, since $\sup f(E) \notin f(E)$, this maximization problem has no solution.

**Example 3.** Let $G = [-1, 1]$, and $f(x)$ is the same as in Example 1. for $x \in G$. It is not difficult to see that (for example, by drawing a picture) the maximization problem $\max_{x \in G} f(x)$ has two solutions: $x = -1$ and $x = 1$.

Next we state two useful general properties of optimization problems.

**Proposition 1.** *Let $f : S \longrightarrow \mathbb{R}$, where $S \subseteq \mathbb{R}^n$; and let $-f$ denote the function whose value at $x$ is $-f(x)$. Then $x \in D$ is a maximizer of $f$ on $D \subseteq S$ if and only if $x$ is a minimizer of $-f$ on $D$; and $y \in D$ is a minimizer of $f$ on $D$ if and only of $y$ is a maximizer of $-f$ on $D$.*

*Proof.* Exercise. ■

Proposition 1 tells us that a generic minimization problem

$$\min_{x \in D} g(x)$$

is equivalent to

$$\max_{x \in D} -g(x);$$

in other words, every minimization problem may be represented as a maximization problem. Therefore, in the remainder of these notes, we devote most of our attention to maximization problems: the tools and results we develop for maximization problems can be easily adapted to minimization problems. Proposition 2 identifies a class of transformations of the objective function that under which the set of solutions remain unchanged.

**Proposition 2.** *Let $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ be a strictly increasing function. Then $x$ is a maximizer of $f$ on $D$ if and only if $x$ is a maximizer of $\varphi \circ f$ on $D$.*

*Proof.* Exercise. ■

Is Proposition 2 still valid if $\varphi$ is only required to be increasing instead of strictly increasing? Why?

**Example 4.** Let $f : [1, 2] \times [1, 2] \longrightarrow \mathbb{R}$ be defined by $f(x, y) = xy$. Since $f$ is strictly increasing in both $x$ and $y$ (how would you show this?), the unique maximizer is $(x, y) = (2, 2)$. Now let $g(z) = \log z$, so

$$g \circ f = \log(xy) = \log x + \log y,$$

whose unique maximizer is also $(2, 2)$.

## 2 Sufficient conditions for existence and uniqueness

Before we study optimization problems in more details, we discuss existence and uniqueness of solutions first. Existence of solutions is a fundamental question in optimization problems: when we cannot guarantee existence, characterizing and/or finding solutions do not make much sense. Theorem 1 is an extremely important and useful result that identifies a (considerably general) set of sufficient conditions for existence of solutions.

**Theorem 1** (The extreme value theorem)**.** *Let $S$ be a nonempty compact subset of $\mathbb{R}^n$, and let $f$ be a continuous real function on $S$. Then there exists $x, y \in S$ with $f(x) = \sup f(S)$ and $f(y) = \inf f(S)$.*

To prove Theorem 1, the key step is that $f(S)$ is a compact subset of $\mathbb{R}$ provided that $f$ is continuous and $S$ is compact.[1] Consequently, the fact that $f(S)$ is compact means it is closed and bounded; so boundedness implies that it cannot be that $\sup f(S) \in \{\infty, -\infty\}$ and/or $\inf f(S) \in \{\infty, -\infty\}$ (in other words, both $\sup f(S)$ and $\inf f(S)$ are well-defined), and then closedness yields that both $\sup f(S)$ and $\inf f(S)$ are contained in $f(S)$.

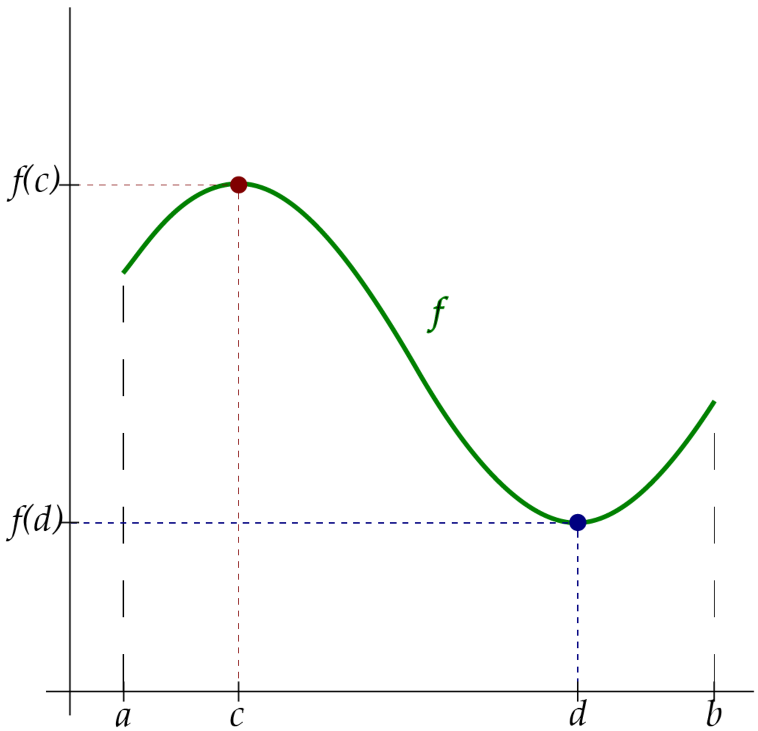


Figure 1: Let $f : [a, b] \longrightarrow \mathbb{R}$ be continuous, then $f$ attains a maximum at $x = c$, and attains a minimum at $x = d$.

It is very important to note that, the extreme value theorem only provides *sufficient* conditions for the existence of optima: if the conditions of the theorem are met, both maxima and minima must exist. But it is silent on what happens when the conditions fail; in particular, these conditions are not necessary: if some of the conditions fail, optima may still exist. In Example 1 and Example 2, the constraint set is not compact, and $f$ fails to attain a maximum; in Example 3, all conditions of the theorem are met, so $f$ attains both maximum and minimum (which is unique, at $x = 0$). In Example 5 below, however, all conditions of the theorem are violated, but both maxima and minima exist.

[1] We omit the proof of this fact; see, for example, Ok (2007), page 222, or Sundaram (1996), page 96. All proofs I mentioned above, however, use some other equivalent (but more general) definitions of compactness, which we do not mention in this class.

**Example 5.** Let $D = (0, \infty)$, and let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 1 - (x-1)^2 & \text{if } 1 \leq x \leq 2, \\ 0 & \text{otherwise.} \end{cases}$$

We see that (draw a picture!) $f$ is not continuous at $x = 1 \in D$, and $D$ is not compact; but $f$ is maximized at $x = 1$, and it attains a minimum at every $x \in (0, 1) \cup [2, \infty)$.

Uniqueness of solutions is another important question. For instance, if we know, before we actually solve a problem, that a unique solution exists, we do not need to worry about multiplicity. More importantly, in many problems the solution depends on some parameters, then uniqueness of solution makes the question of "how solutions change with parameters" way easier to answer. The next result identifies a sufficient condition for uniqueness of solution.

**Theorem 2.** *Let $D$ be a nonempty convex subset of $\mathbb{R}^n$, and $f : D \longrightarrow \mathbb{R}$ is strictly quasiconcave. Then if $f$ attains a maximum on $D$, the solution is unique.*

*Proof.* Suppose there exist $x, y \in D$, $x \neq y$, such that $f(x) = f(y) = \max f(D)$. Since $D$ is convex, fix any $\lambda \in (0, 1)$, we have $\lambda x + (1 - \lambda)y \in D$. But then strict quasiconcavity of $f$ implies that

$$f(\lambda x + (1 - \lambda)y) > \min\{f(x), f(y)\} = \max f(D),$$

a contradiction. ■

Importantly, before we use Theorem 2 to argue that the solution is unique, we have to make sure that a solution indeed exists.

# 3 Interior optima

As the section title suggests, in this section we mainly concern the set of maximizers in the interior of the feasible set. Consider the function $f : [a, b] \longrightarrow \mathbb{R}$ shown in Figure 2. Evidently, the unique maximizer of $f$ is $x''$; and although $x'$ is not a maximizer, but $f$ indeed attains a maximum at $x'$ *amongst the points that are close to it.* The following definition formalizes such points.

**Definition 1** (Local maximizer)**.** Let $D \subseteq S \subseteq \mathbb{R}^n$, and let $f$ be a real-valued function defined on $S$. The point $x^* \in D$ is a **local maximizer** of $f(x)$ on $D$ if there exists $\varepsilon > 0$ such that $f(x^*) \geq f(x)$ whenever $x \in D$ and $d(x, x^*) < \varepsilon$.

A **local minimizer** can be analogously defined. In the remainder of theses notes, we usually refer to a maximizer as a **global maximizer** to emphasize that it is not only a local maximizer. Evidently, every global maximizer (minimizer) is a local maximizer (minimizer).

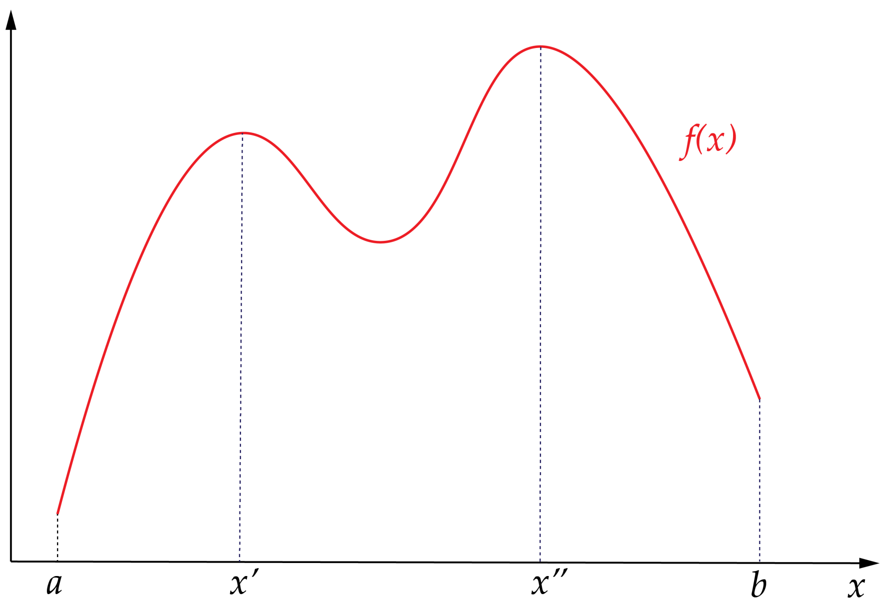


Figure 2: Local maximum and global maximum

In economics and business, as Osborne (2016) points out, we are almost always interested in global maximizers, not merely local maximizers. Nonetheless, as we will soon see, finding local maximizers could be an important step in solving (global) optimization problems.

### 3.1 First-order conditions

Interior local optima have a celebrated but simple property that can be characterized using first derivative, which is usually called *the first-order condition.*

**Proposition 3** (First-order condition, single varible)**.** *Let $I \subseteq \mathbb{R}$ be an interval, and let $f$ be a real-valued function defined on $I$. If $x^* \in \text{int}I$ is a local maximizer or minimizer of $f$, and $f$ is differentiable at $x^*$, then $f'(x^*) = 0$.*

*Proof.* Suppose that $x^*$ is a local maximizer of $f$. Because $x^*$ is in the interior of $I$, for $h > 0$ sufficiently small we have $x^* + h \in I$, so that $f(x^* + h)$ is defined. Thus because $x^*$ is a local maximizer of $f$, for small enough values of $h$, we have $f(x^* + h) \leq f(x^*)$, and hence

$$\frac{f(x^* + h) - f(x^*)}{h} \leq 0.$$

Then by definition of a derivative, the limit of left-hand side of this inequality as $h \longrightarrow 0$ is $f'(x^*)$, hence $f'(x^*) \leq 0$. A symmetric argument using $h < 0$ shows that $f'(x^*) \geq 0$; thus $f'(x^*) = 0$. An analogous argument applies when $x^*$ is a local minimizer of $f$. ∎

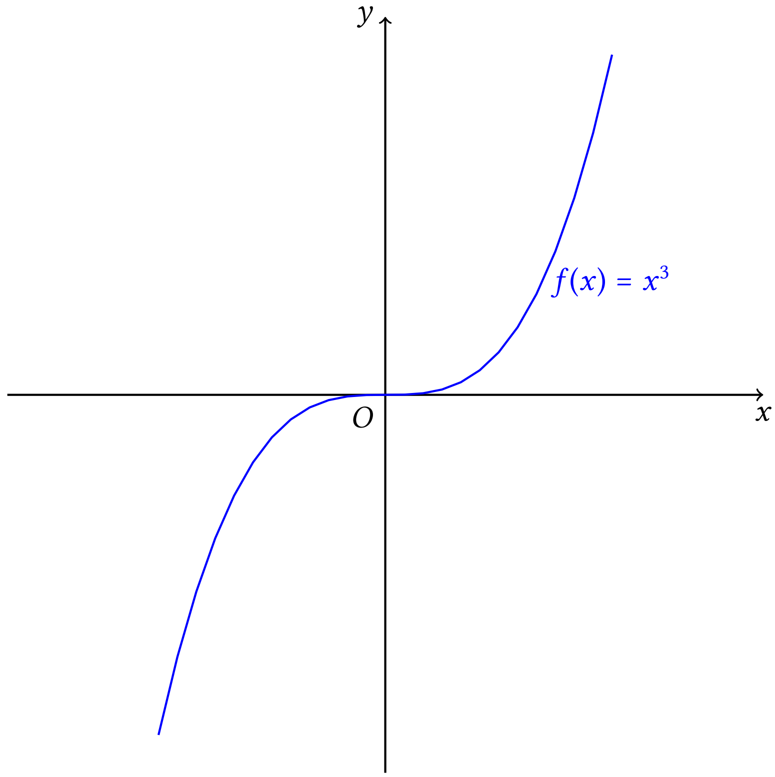


Figure 3: The first-order condition need not to be sufficient: Let $f(x) = x^3$; we have $f'(0) = 0$ but $x = 0$ is neither a local maximizer nor a local minimizer.

Indeed, both $x'$ and $x''$ in Figure 2 are local maximizers, and $f'(x') = f'(x'') = 0$. However, as shown in Figure 3, there may exist points such that the first derivative of $f$ is zero at these points, but $f$ does not achieve local optima there.

Proposition 3 can be generalized to functions with many variables.

**Proposition 4** (First order condition)**.** *Let $S$ be a subset of $\mathbb{R}^n$, and let $f$ be a real-valued function defined on $S$. If $x^* \in \operatorname{int} S$ is a local maximizer or minimizer of $f$, and $f$ is differentiable at $x^*$, then*

$$\frac{\partial f}{\partial x_j}(x^*) = 0$$

*for all $j = 1, \ldots, n$.*

We omit the proof of Proposition 4; it can be easily proved by applying the same argument we used in the proof of Proposition 3 to $n$ partial derivatives. (Try it!)

## 3.2 Solving for optima on a constraint set

Proposition 3 tells us that, among all the points in the interval $I$, only the endpoints (if any) and the points satisfying the first-order condition can be maximizers of a function $f$ of a single variable. For most functions, the first-order condition is only satisfied by a relatively small number of points, so the following "cookbook" procedure to find the maximizers is very useful.

Let $I$ be an interval, and let $f : I \longrightarrow \mathbb{R}$. If the problem $\max_{x \in I} f(x)$ has solutions, they may be found as follows:

- Find all $x \in I$ such that $f'(x) = 0$, and calculate the values of $f$ at each such point.
- Find the values of $f$ at the endpoints, if any, of $I$.
- Among all the points you have found, the ones at which the value of $f$ is largest are the maximizers of $f$.

The variant of this procedure in which the last step involves choosing the points $x$ at which $f(x)$ is smallest may be used to solve the analogous minimization problem.

**Example 6.** Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined as $f(x) = x^2$; consider the problem

$$\max_{x \in [-1,1]} f(x).$$

Since $f(x)$ is continuous, and $[-1, 1]$ is compact, Theorem 1 implies that the maximization problem has a solution. It is not difficult to see that the only point satisfying the first-order condition is $x = 0$, and the value of the function at this point is f(0) = 0. The values of $f$ at the endpoints of $[-1, 1]$ are $f(-1) = 1$ and $f(1) = 1$. Thus the global maximizer of $f$ on $[-1, 1]$ are $x = 1$ and $x = -1$. In fact, $x = 0$ is the unique global minimizer.

And similarly for functions defined on $S \subseteq \mathbb{R}^n$, Proposition 4 implies that only points that can be global maximizers are either boundary points of the set $S$, or those satisfying the first-order conditions.

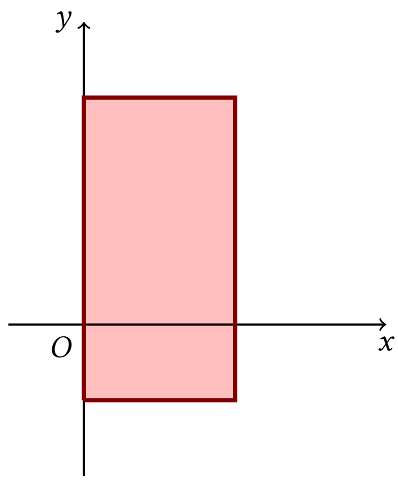


Figure 4: The constraint set of the problem Example 7 (pink) and its boundary (maroon).

Let $S$ be a subset of $\mathbb{R}^n$, and let $f : S \longrightarrow \mathbb{R}$. If the problem

$$\max_{x \in S} f(x)$$

has solutions, they may be found as follows:

- Find all points in $\text{int}S$ satisfying the first-order conditions.
- Among the points in $S \setminus \text{int}S$, find those at which the value of $f$ is largest.
- Among all the points you have found, the ones at which the value of $f$ is largest are the maximizers of $f$.

By replacing every "largest" by "smallest" in the procedure above, we are able locates the solutions of the analogous minimization problem.

**Example 7.** Let $g : \mathbb{R}^2 \longrightarrow \mathbb{R}_+$ be defined as $g(x) = (x-1)^2 + (y-1)^2$; consider the problem

$$\max_{x \in [0,2],\ y \in [-1,3]} g(x, y).$$

This problem satisfies the conditions of Theorem 1, so it has at least one solution. The first-order conditions are

$$2(x-1) = 0$$
$$2(y-1) = 0$$

and the lone point in $\mathbb{R}^2$ satisfying these conditions is $(1, 1)$, and it is in the constraint set. The value of the objective function at this point is 0.

Now consider the behavior of the objective function on the boundary of the constraint set, which is a rectangle; see Figure 4.

- Let us consider the left boundary of the rectangle in Figure 4 first, where $x = 0$ and $-1 \leq y \leq 3$. The value of the objective function on that line segment is $1 + (y - 1)^2$. The problem of finding $y$ to maximize this function subject to $-1 \leq y \leq 3$ satisfies the conditions of Theorem 1, and thus has a solution.

  The first-order condition is $2(y - 1) = 0$, which has a unique solution $y = 1 \in [-1, 3]$. The value of the objective function at this point is 1. On the boundary of which is in the constraint set. The value of the objective function at this point is 1. On the boundary of the set $\{(0, y) : -1 \leq y \leq 3\}$, namely at the points (0,-1) and $(0, 3)$, the value of the objective function is 5. Thus on this part of the boundary, the points $(0, -1)$ and $(0, 3)$ are the only candidates for a solution of the original problem.

- A similar analysis leads to the conclusion that the points (2,-1) and (2,3) are the only candidates for a maximizer on the right boundary of the rectangle where $x = 2$ and $-1 \leq y \leq 3$. Moreover, the points $(0, -1)$ and $(2, -1)$ are the only candidates for a maximizer on the top part of the boundary for which $0 \leq x \leq 2$ and $y = -1$, and the points (0,3) and (2,3) are the only candidates for a maximizer on the part of the boundary for which $0 \leq x \leq 2$ and $y = 3$.

- Therefore, the value of the objective function at all these candidates for a solution on the boundary of the constraint set is 5.

Finally, comparing the values of the objective function at the candidates for a solution that are (a) interior to the constraint set (namely $(1, 1)$) and (b) on the boundary of the constraint set, we conclude that the problem has four solutions, $(0, -1)$, $(0, 3)$, $(2, -1)$, and $(2, 3)$. The maximal value of the function on the constraint set is 5.

### 3.3 Second-order conditions

Observe that Proposition 3 and Proposition 4 do not distinguish between local maxima and local minima. A set of results, usually called *the second-order conditions*, by examining the behavior of second derivatives at an optimum, not only allow us to obtain such a distinction, but also provide sufficient conditions that identify specifit points as being local optima. We do not present these results in these notes; interested readers are directed to Osborne (2016), Section 5.2, or Sundaram (1996), Section 4.3.

## 3.4 First-order conditions and global optima

If $f$ is a concave differentiable function of single variable, the first-order condition becomes sufficient, and more importantly, points satisfying the first-order condition are guaranteed to be *global* maximizers of $f$. This is because a concave and differentiable function has the following property: for every point $x$, the graph of a $f$ lies on or below the tangent to $f$ at $x$. Thus if $x^*$ satisfies the first-order condition, so $f'(x^*) = 0$, $x^*$ must be a global maximizer of $f$ (see Figure 5 for an illustration). Similarly, a differentiable convex function lies on or above any of its tangents, so points satisfying the first-order condition are guaranteed to be *global* minimizers of $f$. We thus have the following result.

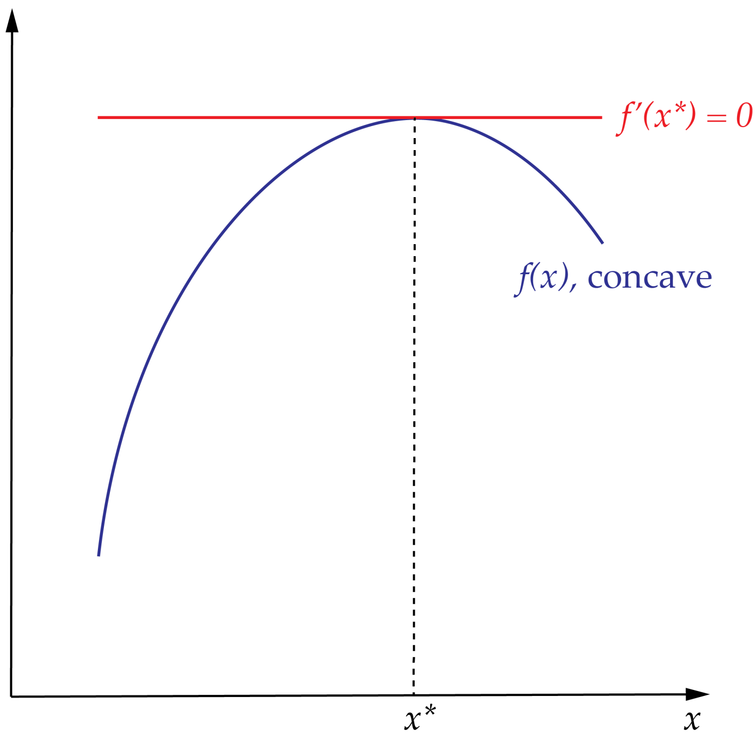


Figure 5: The first-order condition identifies global maximizers for concave differentiable functions.

**Proposition 5.** *Let $I \subseteq \mathbb{R}$ be an interval, and let $f$ be a real-valued differentiable function defined on $I$. If $x^* \in \text{int} I$, then*

- *if $f$ is concave then $x^*$ is a global maximizer of $f$ in $I$ if and only if $f'(x^*) = 0$; and*
- *if $f$ is convex then $x^*$ is a global minimizer of $f$ in $I$ if and only if $f'(x^*) = 0$.*

*Proof.* The "only if" direction follows from Proposition 3; it remains to prove "if". Recall that, if $f$ is a differentiable function defined on an interval $I$, then $f$ is concave if and only if

$$f(x) - f(y) \leq f'(y)(x - y)$$

for all $x, y \in I$. Then because $f$ is concave on $I$ and $f'(x^*) = 0$, we must have

$$f(x) - f(x^*) \leq 0$$

for all $x \in I$; so $x$ is a global maximizer of $f$ in $I$. Similarly, if $f$ is convex then $f(x') \geq f(x)$ for all $x' \in I$. ■

**Example 8.** Let $h : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $h(x) = -x^2$. Consider the problem

$$\max_{x \in [-1,1]} h(x).$$

Observe that the objective function $h$ is (strictly) concave; solve the first-order condition

$$h'(x^*) = -2x^*,$$

we get that $x^* = 0$. By Proposition 5, the global maximizer of $h$ is $x^* = 0$, which is in $[-1, 1]$ and is thus the unique solution of the problem.

**Example 9.** A firm produces a single product using a single input. The unit price of the input is $w$ and the unit price of output is $p$. The firm's output from $x$ units of the input is $\sqrt{x}$. The firm's profit maximization problem is

$$\max_{x \geq 0} p\sqrt{x} - wx.$$

The first derivative of the objective function is $(1/2)px^{-1/2} - w_1$ and the second derivative is $-(1/4)px^{-3/2}$, which is less than zero for all $x \geq 0$; hence, the objective function is concave. Thus the solution of this problem can be obtained by solving $(1/2)p(x^*)^{-1/2} - w = 0$, so that $x^* = (p/2w)^2$.

Similar to Proposition 3, Proposition 5 can also be generalized to functions with many variables.

**Proposition 6.** *Let $T$ be a convex subset of $\mathbb{R}^n$, and let $f : T \longrightarrow \mathbb{R}$ be differentiable. If $x^* \in \operatorname{int} T$, then*

- *if $f$ is concave then $x^*$ is a global maximizer of $f$ in $T$ if and only if*

$$\frac{\partial f}{\partial x_j}(x^*) = 0$$

*for all* $j = 1, \dots, n$*; and*

- *if* $f$ *is convex then* $x^*$ *is a global minimizer of* $f$ *in* $T$ *if and only if*

$$\frac{\partial f}{\partial x_j}(x^*) = 0$$

*for all* $j = 1, \dots, n$.

# 4 Detour: The implicit function theorem

In many models in economics and business, the optimal choice or equilibrium value of a variable $x$ can be expressed as the solution of an equation

$$f(x, \alpha) = 0, \tag{1}$$

where $f$ is a function, and $\alpha$ is a parameter.

**Example 10.** Consider the following optimization problem

$$\max_{x \in A} F(x, \alpha),$$

where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a strictly concave and differentiable function, $A$ is a closed interval, and $\alpha$ is a parameter. Then the first-order condition is sufficient, and if we let

$$f = \frac{\partial F}{\partial x},$$

the optimal choice of $x$ is the solution to Equation (1).

In such cases, we sometimes want to answer the question that how the optimal choice or equilibrium value of $x$ depends on the parameter $\alpha$. For example, does it increase or decrease when the value of the parameter increases?

Sometimes it is easy to write the optimal choice or equilibrium value of $x$, denote by $x^*$, explicitly as a function of parameter $\alpha$; for example, if

$$f(x, \alpha) = 2\alpha - x = 0,$$

it is clear that $x^* = 2\alpha$, so it is strictly increasing in $\alpha$. In many other cases, however, we cannot write down explicit formulae like this. Fortunately, with the help of the celebrated implicit function theorem, under some conditions, we are still able to answer the question: the theorem states that $x^*(\alpha)$ exists, and more importantly, its derivative is continuous and can be derived from the expression of $f(x, a)$.

**Definition 2** (Level curves)**.** Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$. The set

$$\mathcal{C}_f(k) = \{(x, y) \in \mathbb{R}^2 : f(x, y) = k\},$$

which is the set of pairs $(x, y) \in \mathbb{R}^2$ such that $f(x, y) = k$ is called the **level curve** of $f$ for the value $k$.

**Example 11.** Let $h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined by $h(x, y) = x^2 + y^2$. The level curve of $h$ for the value 1 is the set

$$\mathcal{C}_h(1) = \{(x, y) \in \mathbb{R}^2 : h(x, y) = x^2 + y^2 = 1\},$$

which is drawn in Figure 6.

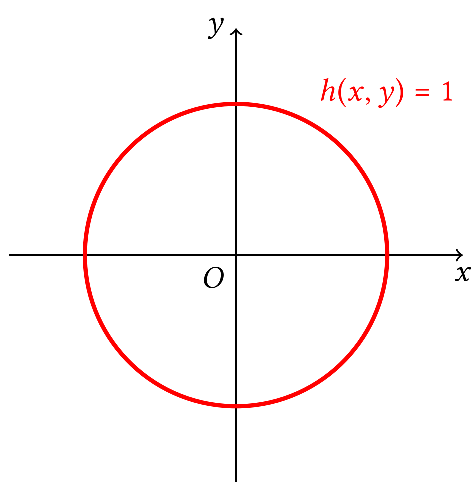


Figure 6: The level curve of the function $h(x, y) = x^2 + y^2$ for the value 1.

**Example 12.** The unit circle $C_U = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$ can be specified as the level curve $\mathcal{C}_h(1)$ of the function $h(x, y) = x^2 + y^2$, which is shown in Figure 6. Around point $a$, $y$ can be expressed as a function $g(x)$. In this example this function can be written explicitly as $g_1(x) = \sqrt{1 - x^2}$ around point $a$. In many cases no such explicit expression exists, but one can still refer to the implicit function $y = g(x)$. No such function exists around point $b$. (Why?[2])

**Definition 3** (Continuously differentiable functions)**.** Let $f : U \longrightarrow \mathbb{R}$.

[2] *Hint.* Take $\varepsilon > 0$ small, and think about what happens to the corresponding $y$'s when we let $x = 1 + \varepsilon$ and $x = 1 - \varepsilon$, respectively.

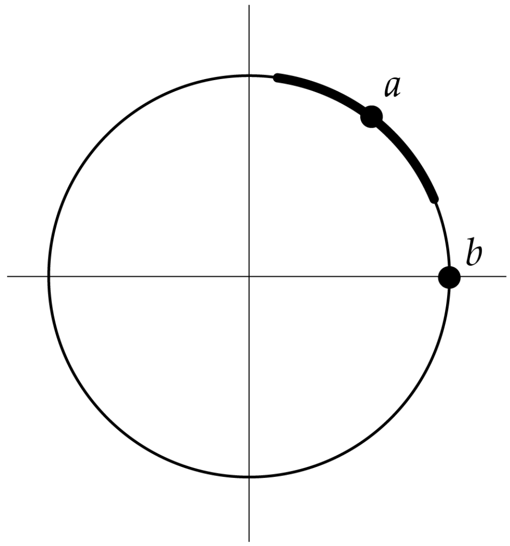


Figure 7: The unit circle $C_U = \{(x, y) \in \mathbb{R}^2 \,:\, x^2 + y^2 = 1\}$.

(1) If $U$ is an open interval, $f$ is said to be **continuously differentiable** if the derivative $f'$ exists and is itself a continuous function.

(2) If $U$ is an open subset of $\mathbb{R}^n$, $f$ is said to be **continuously differentiable** if all its partial derivatives exist and are continuous.

**Theorem 3** (Implicit function theorem)**.** *Let $S$ be an open subset of $\mathbb{R}^2$. Suppose $F \,:\, S \longrightarrow \mathbb{R}$ is a continuously differentiable function defining a curve $F(x, y) = 0$; let $(x_0, y_0)$ be a point on the curve. If*

$$\left.\frac{\partial F}{\partial y}\right|_{(x_0,y_0)} \neq 0, \tag{2}$$

*then there exists a continuously differentiable function $g$ defined on an open interval $I$ containing $x_0$ such that $y_0 = g(x_0)$, and at any $x \in I$ we have*

$$F(x, g(x)) = 0, \tag{3}$$

*and*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = g'(x) = -\frac{\partial F/\partial x}{\partial F/\partial y}. \tag{4}$$

We omit the proof of Theorem 3 and provide an heuristic argument for Equation (4) only.[3]

[3]See Simon and Blume (1994), page 344, Theorem 15.3 for a nice sketch of the proof. For those who know some differential equations, you are encouraged to prove Theorem 3 using the Picard-Lindelöf theorem on the existence and uniqueness of a solution of a general ordinary differential equation (See, for example, Ok (2007), page 188). For a more general statement of the theorem and its proof, see Rudin (1976), page 224, Theorem 9.28.

Differentiate both sides of Equation (3), by a version of the chain rule we have

$$\frac{\partial F}{\partial x}(x, g(x)) + \frac{\partial F}{\partial y}(x, g(x))g'(x) = 0.$$

Then so long as Equation (2) holds, for $x$ sufficiently close to $x_0$, we have $\partial F(x, g(x))/\partial y \neq 0$. A rearrangement of the above equality yields Equation (4).

**Example 13.** Consider the equation

$$G(x, y) = x^2 - 3xy + y^3 - 7 = 0 \tag{5}$$

around the point $(x_0, y_0) = (4, 3)$. One can compute that

$$\begin{aligned} \frac{\partial G}{\partial x} &= 2x - 3y, \\ \frac{\partial G}{\partial y} &= -3x + 3y^2; \end{aligned}$$

then we have

$$\frac{\partial G}{\partial x}(4, 3) = -1 \quad \text{and} \quad \frac{\partial G}{\partial y}(4, 3) = 15.$$

By Theorem 3, $G(x, y)$ does indeed define $y$ as a continuously differentiable function of $x$ around $(x_0, y_0) = (4, 3)$, and

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x_0) = -\frac{\frac{\partial G}{\partial x}(x_0, y_0)}{\frac{\partial G}{\partial y}(x_0, y_0)} = \frac{1}{15}.$$

# 5 Equality constraints

Let $S$ be an open subset of $\mathbb{R}^n$. Let $f : S \longrightarrow \mathbb{R}$ and $g_j : S \longrightarrow \mathbb{R}$, $j = 1, \dots, m$ be continuously differentiable functions. Consider an optimization problem of the form

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \text{subject to} \quad & g_j(x) = 0 \text{ for } j = 1, \dots, m \end{aligned} \tag{6}$$

with $m \leq n$.

## 5.1 A special case

To grasp some intuition, we study a special case of problem (6) first: let $f$ be a function of two variables (that is, $n = 2$) and $m = 1$, so there is only one equality constraint. That is, we work

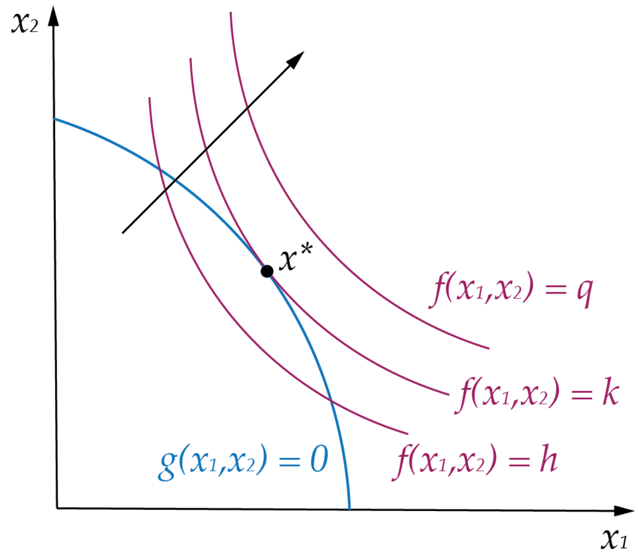


Figure 8: Maximizing a function of two variables subject to an equality constraint.

with a problem of the form

$$\max_{x \in S} \quad f(x) \qquad (7)$$
$$\text{subject to} \quad g(x) = 0.$$

It is usually convenient to represent a function defined on a subset of $\mathbb{R}^2$ by a family of level curves. In particular, for problem (7), we can represent the constraint $g(x) = 0$ in the $(x_1, x_2)$–plane as the level curve

$$\mathcal{C}_g(0) = \{x \in \mathbb{R}^2 \,:\, g(x) = 0\},$$

which is the blue curve in Figure 8. Assume that $f$ is increasing to the northeast, so in Figure 8 we have $h < k < q$.

Assume further that the functions $f$ and $g$ are differentiable, then we see from Figure 8 that at a solution $x^*$ of problem (7), the level curve $\mathcal{C}_g(0)$ representing the constraint is tangent to a level curve of $f$, $\mathcal{C}_f(k)$. Consequently, the two level curves have the same slope at $x^*$; by the implicit function theorem, we know that

$$-\frac{\frac{\partial f}{\partial x_1}(x^*)}{\frac{\partial f}{\partial x_2}(x^*)} = -\frac{\frac{\partial g}{\partial x_1}(x^*)}{\frac{\partial g}{\partial x_2}(x^*)}.$$

Now we introduce a new variale $\lambda \in \mathbb{R}$, which is defined as

$$\lambda = -\frac{\frac{\partial f}{\partial x_1}(x^*)}{\frac{\partial g}{\partial x_1}(x^*)} = -\frac{\frac{\partial f}{\partial x_2}(x^*)}{\frac{\partial g}{\partial x_2}(x^*)}, \qquad (8)$$

provided that both $\frac{\partial g}{\partial x_1}(x^*)$ and $\frac{\partial g}{\partial x_2}(x^*)$ are nonzero. One might be tempted to argue that introducing a new variable merely complicates the problem, but as we will see shortly, it is in fact a clever step that allows the condition for a maximum to be expressed in an appealing way.

Therefore, at $x^*$, both Equation (8) and the constraint $g(x^*) = 0$ must hold. These conditions can be written as the following system of equations:

$$\begin{aligned} \frac{\partial f}{\partial x_1}(x^*) + \lambda \frac{\partial g}{\partial x_1}(x^*) &= 0, \\ \frac{\partial f}{\partial x_2}(x^*) + \lambda \frac{\partial g}{\partial x_2}(x^*) &= 0, \\ g(x^*) &= 0. \end{aligned}$$

The three equations can be viewed conveniently as the first-order conditions of the **Lagrangian**

$$\mathcal{L}(x, \lambda) = f(x) + \lambda g(x)$$

with respect to $x$, $y$ and $\lambda$, respectively. We usually call the real number $\lambda$ the **Lagrangian multiplier** of the problem. We can think of the term $\lambda g(x)$ as the punishment of violating the constraint.

The discussion above is summarized in Proposition 7. Intuitively, this result says that we can set penalties for violating the equality constraint in such a way that at the maximum the constraint is exactly satisfied.

**Proposition 7** (Equality constraints, a special case)**.** *Let $S$ be an open subset of $\mathbb{R}^2$, and let $f : S \longrightarrow \mathbb{R}$ and $g : S \longrightarrow \mathbb{R}$ be continuously differentiable functions. If $x^* = (x_1^*, x_2^*) \in S$ and it is a solution to problem*

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \text{subject to} \quad & g(x) = 0 \end{aligned}$$

*and suppose also that*

$$\textit{either} \quad \frac{\partial g}{\partial x_1}(x^*) \neq 0 \quad \textit{or} \quad \frac{\partial g}{\partial x_2}(x^*) \neq 0.$$

*Then there exists a unique $\lambda^* \in \mathbb{R}$ such that $x^*$ satisfies the first-order conditions*

$$\frac{\partial \mathcal{L}}{\partial x_1}(x^*, \lambda^*) = \frac{\partial f}{\partial x_1}(x^*) + \lambda^* \frac{\partial g}{\partial x_1}(x^*) = 0;$$
$$\frac{\partial \mathcal{L}}{\partial x_2}(x^*, \lambda^*) = \frac{\partial f}{\partial x_2}(x^*) + \lambda^* \frac{\partial g}{\partial x_2}(x^*) = 0;$$
$$\frac{\partial \mathcal{L}}{\partial \lambda}(x^*, \lambda^*) = g(x^*) = 0.$$

**Example 14.** Let $f(x_1, x_2) = x_1 x_2$, and let $g(x_1, x_2) = x_1 + 4x_2 - 8$. Consider the following maximization problem:

$$\max_{x_1, x_2 \in \mathbb{R}^2} \quad f(x_1, x_2)$$
$$\text{s.t.} \quad g(x_1, x_2) = 0.$$

Observe that $\nabla g(x_1, x_2) = (1, 4)$ for all $(x_1, x_2) \in \mathbb{R}^2$, so we can use Proposition 7. Form the Lagrangian

$$\mathcal{L}(x_1, x_2, \lambda) = x_1 x_2 + \lambda (x_1 + 4x_2 - 8);$$

by Proposition 7, the necessary conditions are

$$\frac{\partial \mathcal{L}}{\partial x_1} = x_2 + \lambda = 0 \tag{9}$$
$$\frac{\partial \mathcal{L}}{\partial x_2} = x_1 + 4\lambda = 0 \tag{10}$$
$$x_1 + 4x_2 - 8 = 0. \tag{11}$$

We know from (9) and (10) that

$$-\lambda = x_2 = \frac{1}{4} x_1; \tag{12}$$

then by (12) and (11), we have

$$4x_2 + 4x_2 = 8,$$

so $x_2 = 1$. Again by (12), we know that $x_1 = 4$ and $\lambda = -1$. Proposition 7 implies that the only candidate for a solution to the problem is $(x_1, x_2) = (4, 1)$. In other words, if the problem has a solution, it must be $(x_1, x_2) = (4, 1)$.

## 5.2 Interpreting Lagrangian multipliers

We continue working with problem (7), but replace the constraint by

$$g(x_1, x_2) = \eta,$$

and take $\eta$ as a variable. We can think of $\eta$ being either positive or negative and close to zero. Then the solutions of the new problem should be dependent on $\eta$; in other words, a solution can be written as $x^*(\eta) = (x_1^*(\eta), x_2^*(\eta))$. Form the Lagrangian

$$\mathcal{L} = f(x) + \lambda[g(x) - \eta],$$

and by Proposition 7, the first-order conditions[4]

$$\frac{\partial \mathcal{L}}{\partial x_1}(x^*(\eta)) = \frac{\partial f}{\partial x_1}(x^*(\eta)) + \lambda^*(\eta)\frac{\partial g}{\partial x_1}(x^*(\eta)) = 0 \tag{13}$$

$$\frac{\partial \mathcal{L}}{\partial x_2}(x^*(\eta)) = \frac{\partial f}{\partial x_2}(x^*(\eta)) + \lambda^*(\eta)\frac{\partial g}{\partial x_2}(x^*(\eta)) = 0 \tag{14}$$

must hold. Define

$$f^*(\eta) = f(x^*(\eta)),$$

and assume that both $x_1^*(\cdot)$ and $x_2^*(\cdot)$ are differentiable. Differentiate $f(x^*(\eta))$ with respect to $\eta$, by the chain rule we have

$$\begin{aligned} f^{*\prime}(\eta) &= \frac{\partial f}{\partial x_1}(x^*(\eta))x_1^{*\prime}(\eta) + \frac{\partial f}{\partial x_2}(x^*(\eta))x_2^{*\prime}(\eta) \\ &= -\lambda^*(\eta)\left[\frac{\partial g}{\partial x_1}(x^*(\eta))x_1^{*\prime}(\eta) + \frac{\partial g}{\partial x_2}(x^*(\eta))x_2^{*\prime}(\eta)\right], \end{aligned}$$

where the second equality follows from (13) and (14). Differentiate the constraint on both sides, again by the chain rule, we have

$$\frac{\partial g}{\partial x_1}(x^*(\eta))x_1^{*\prime}(\eta) + \frac{\partial g}{\partial x_2}(x^*(\eta))x_2^{*\prime}(\eta) = 1,$$

which implies that

$$\boxed{\lambda^*(0) = -f^{*\prime}(0).} \tag{15}$$

That is, the value of the Lagrange multiplier at the solution of the problem is equal to the inverse of the instantaneous rate of change in the maximal value of the objective function as the constraint

[4] To save notation, we suppress the dependence of $\mathcal{L}$ on $\lambda$ here.

is relaxed.

### 5.3 The general case

Proposition 7 can be generalized to study problem (6) we stated at the beginning of this section. The Lagrangian for that problem is

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x);$$

that is, there is one Lagrangian multiplier for *each* constraint.

**Definition 4** (Jacobian matrix)**.** For $j = 1, \ldots, m$, let $g_j : \mathbb{R}^n \longrightarrow \mathbb{R}$ be differentiable. The **Jacobian matrix** of $(g_1, \ldots, g_m)$ at the point $x \in \mathbb{R}^n$ is

$$J(x) = \begin{pmatrix} \frac{\partial g_1}{\partial x_1}(x) & \frac{\partial g_1}{\partial x_2}(x) & \cdots & \frac{\partial g_1}{\partial x_n}(x) \\ \frac{\partial g_2}{\partial x_1}(x) & \frac{\partial g_2}{\partial x_2}(x) & \cdots & \frac{\partial g_2}{\partial x_n}(x) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1}(x) & \frac{\partial g_m}{\partial x_2}(x) & \cdots & \frac{\partial g_m}{\partial x_n}(x) \end{pmatrix}.$$

**Theorem 4** (Necessity, equality constraints)**.** *Let $S$ be an open subset of $\mathbb{R}^n$, and let $f : S \longrightarrow \mathbb{R}$ and $g_j : S \longrightarrow \mathbb{R}$, $j = 1, \ldots, m$ be continuously differentiable functions with $m \leq n$. If $x^* \in S$ and it is a solution to problem*

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \text{subject to} \quad & g_j(x) = 0 \;\; \text{for} \;\; j = 1, \ldots, m \end{aligned}$$

*and suppose the Jacobian matrix of $(g_1, \ldots, g_m)$ at the point $x^*$ has $m$ linearly independent columns (that is, the rank of the matrix is $m$). Then there exists a unique $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \in \mathbb{R}^m$ such that $x^*$ satisfies the first-order conditions*

$$\frac{\partial \mathcal{L}}{\partial x_k}(x^*, \lambda^*) = \frac{\partial f}{\partial x_k}(x^*) + \sum_{i=1}^{m} \lambda_j^* \frac{\partial g_j}{\partial x_k}(x^*) = 0 \tag{16}$$

*for $k = 1, \ldots, n$, and*

$$\frac{\partial \mathcal{L}}{\partial \lambda_j}(x^*, \lambda^*) = g_j(x^*) = 0 \tag{17}$$

*for $j = 1, \ldots, m$.*

See Appendix A for the definition of linear independence and rank of a matrix. For a proof of Theorem 4, see Section 5.6 of Sundaram (1996), or Simon and Blume (1994), page 478–480.

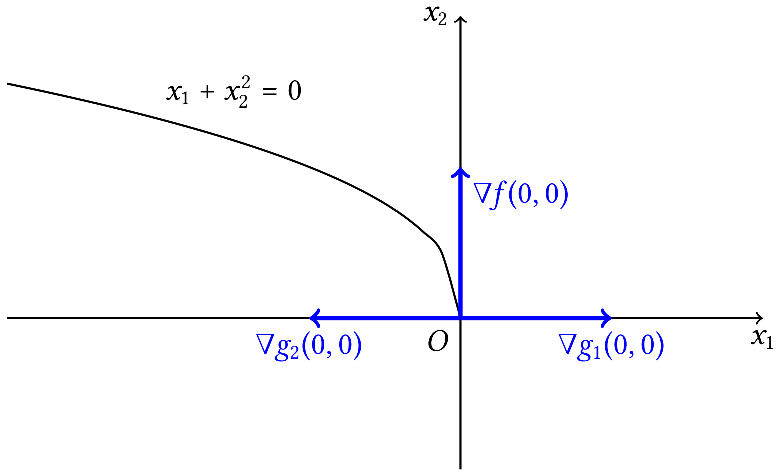


Figure 9: No multipliers exist

The assumption on the rank of the Jacobian matrix, as illustrated by Example 15, is required to make sure that there exist $\lambda^* \in \mathbb{R}^m$ such that the gradient of $f$ can be expressed as a linear combination of gradients of the constraint functions at a solution $x^*$. Roughly, we would like to find $(x^*, \lambda^*) \in \mathbb{R}^{n+m}$, so there are $n + m$ variables; and from necessary conditions (16) and (17) we get $n + m$ equations. When the maximal number of linearly independent columns of the Jacobian falls to $q < m$, $q - m$ equations in (16) and (17) do not provide incremental information about $\lambda_j$'s.

**Example 15.** Let $f(x_1, x_2) = x_2$, $g_1(x_1, x_2) = x_1$, and $g_2(x_1, x_2) = -x_1 - x_2^2$. Consider the problem

$$\begin{aligned} \max_{(x_1,x_2)\in\mathbb{R}^2} \quad & f(x_1, x_2) \\ \text{subject to} \quad & g_1(x_1, x_2) = 0 \\ & g_2(x_1, x_2) = 0. \end{aligned}$$

Clearly, the only point in $\mathbb{R}^2$ that satisfies the constraints is $(0, 0)$; so $(0, 0)$ must be the unique solution of the problem. Now

$$\nabla f(x_1, x_2) = (0, 1), \quad \nabla g_1(x_1, x_2) = (1, 0), \quad \text{and} \quad \nabla g_2(x_1, x_2) = (-1, -2x_2);$$

then $\nabla g_2(0, 0) = (-1, 0)$. By definition, the Jacobian matrix at $(0, 0)$ is

$$J(0,0) = \begin{pmatrix} 1 & 0 \\ -1 & 0 \end{pmatrix}$$

whose rank is $1 < 2$. As shown in Figure 9, there does not exist $(\lambda_1, \lambda_2)$ such that

$$\nabla f(0,0) + \lambda_1 \nabla g_1(0,0) + \lambda_2 \nabla g_2(0,0) = 0;$$

in other words, although $(0,0)$ solves the original problem, it does not satisfy Equation (16), namely the first-order condition of the Lagrangian.

# 6 Inequality constraints

As in Section 5, let $S$ be an open subset of $\mathbb{R}^n$, and let $f$ and $g_j$, $j = 1, \ldots, m$, be real-valued continuously differentiable functions defined on $S$. In this section, we turn to optimization problems of the form

$$\begin{aligned} \max_{x \in S} \quad & f(x), \\ \text{subject to} \quad & g_j(x) \geq 0 \text{ for } j = 1, \ldots, m. \end{aligned} \tag{18}$$

Note that problem (6) we studied in Section 5 is really a special case of problem (18).

## 6.1 Necessity

For expository convenience, assume $m = 1$, so there is only one constraint $g(x) \geq 0$, where $g : S \longrightarrow \mathbb{R}$ is a differentiable function. Now the problem simplifies to

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \text{subject to} \quad & g(x) \geq 0. \end{aligned} \tag{19}$$

For any $x \in S$ satisfying the inequality constraint, we say that the constraint is **binding** if $g(x) = 0$, and **slack** (or **nonbinding**) if $g(x) > 0$.

As in Section 5, define the Lagrangian $\mathcal{L}$ by

$$\mathcal{L}(x, \lambda) = f(x) + \lambda g(x).$$

For any $x^* \in S$ that solves problem Equation (19), there are two possibilities:

- $g(x^*) = 0$, so the constraint is binding at $x^*$; then if the constraint "moves" by a small amount in the sense that it becomes
$$g(x) \geq \eta$$

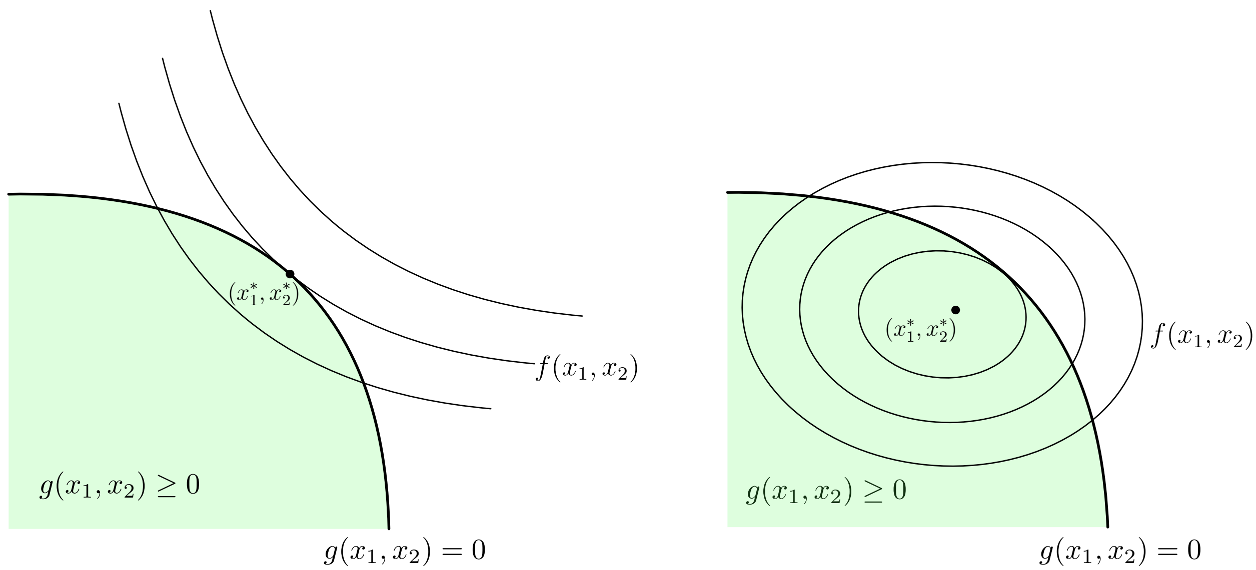


Figure 10: The inequality constraint is binding (left panel) and slack (right panel).

for some $\eta$ could be positive or negative but close to zero, the $x^*$ might no longer be optimal.

- $g(x^*) > 0$, so the constraint is slack at $x^*$; then the constraint "moves" by a small amount, the solution is unaffected.

Therefore, if $g(x^*) = 0$, we are back to the situation we faced in Section 5: under some regularity conditions, we have[5]

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = \left( \frac{\partial \mathcal{L}}{\partial x_1}(x^*, \lambda^*), \frac{\partial \mathcal{L}}{\partial x_2}(x^*, \lambda^*), \ldots, \frac{\partial \mathcal{L}}{\partial x_n}(x^*, \lambda^*) \right) = 0.$$

Note that, in this case, we must have $\lambda^* \geq 0$. Suppose to the contrary that $\lambda^* < 0$, then we know from the discussion in Section 5.2 that letting $g(x) = \eta$ for some $\eta$ positive but close to zero raises the maximal value of $f$ (recall Equation (15)). That is, moving $x^*$ *inside* the constraint raises the value of $f$, contradicting the fact that $x^*$ is the solution of the problem.

And if $g(x^*) > 0$, the solution is in the interior of the constraint set, hence we can appeal to the results we developed in Section 3 to get that

$$\nabla f(x^*) = \left( \frac{\partial f}{\partial x_1}(x^*), \frac{\partial f}{\partial x_2}(x^*), \ldots, \frac{\partial f}{\partial x_n}(x^*) \right) = 0.$$

In this case, the value of $\lambda$ does not enter the conditions, so we can choose any value for it. Given the interpretation of the Lagrangian multiplier $\lambda$, setting $\lambda^* = 0$ makes sense. This assumption

[5]The notation $\nabla_x \mathcal{L}(x^*, \lambda^*)$ means the vector of partial derivatives of the function $\mathcal{L}$ with respect to the coordinates of the vector $x$ evaluated at $(x^*, \lambda^*)$.

implies that

$$\frac{\partial \mathcal{L}}{\partial x_k}(x^*, \lambda^*) = \frac{\partial f}{\partial x_k}(x^*) = 0$$

for all $k = 1, \dots, n$.

Thus, in both cases, we have $\nabla_x \mathcal{L}(x^*, \lambda^*) = 0$, $\lambda^* \geq 0$, and $g(x^*) \geq 0$; and when the constraint is binding we have $g(x^*) = 0$, while $\lambda^* = 0$ holds when the constraint is slack. Then since the product of two numbers is zero if and only if at least one of them is zero, we can combine the two cases and get the following set of necessary conditions:

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0 \tag{20}$$

$$\lambda^* \geq 0 \tag{21}$$

$$g(x^*) \geq 0 \tag{22}$$

$$\lambda^* g(x^*) = 0 \tag{23}$$

that is, if $x^*$ is a solution to problem (19), it must satisfy conditions (20) to (23). Equation (23) suggests that either

(i) $\lambda^* = 0$ and $g(x^*) \geq 0$, or

(ii) $\lambda^* \geq 0$ and $g(x^*) = 0$;

(i) and (ii) are called **complementary slackness conditions**; note that it does not rule out the possibility that both $\lambda^* = 0$ and $g(x^*) = 0$.

For a problem with many constraints (that is, $m \geq 1$), we may assign a Lagrangian multiplier to each of the constraints to get the Lagrangian

$$\mathcal{L} = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x);$$

Then we can apply our argument to each of the constraints to get an analogy of Theorem 4 for problems with inequality constraint. Before stating the result, we introduce some regularity conditions, which are usually called **constraint qualifications (CQ)**, which guarantee necessity.

We summarize three most popular constraint qualifications:

(A) Let $\left(g_1^B, \ldots, g_q^B\right)$ be the constraints that are binding at $x^*$, with $q \leq m$. Then $q \leq n$, and the Jacobian matrix of $\left(g_1^B, \ldots, g_q^B\right)$ at the point $x^*$ has $q$ linearly independent columns.

(B) $g_j$ is convex for $j = 1, \ldots, m$.

(C) $g_j$ is concave for $j = 1, \ldots, m$ and **Slater's condition** holds: there exists $\hat{x} \in S$ such that $g_j(\hat{x}) > 0$ for all $j = 1, \ldots, m$.

**Theorem 5** (Necessity, inequality constraints)**.** *Let $S$ be an open subset of $\mathbb{R}^n$, and let $f : S \longrightarrow \mathbb{R}$ and $g_j : S \longrightarrow \mathbb{R}$, $j = 1, \ldots, m$, be continuously differentiable functions. If $x^* \in S$ and it is a solution to problem*

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \geq 0 \text{ for } j = 1, \ldots, m \end{aligned}$$

*and suppose at least one of the CQs listed above is satisfied. Then there exists $\lambda^* = (\lambda_1^*, \ldots, \lambda_m^*) \in \mathbb{R}_+^m$ such that*

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0, \tag{24}$$

$$g_j(x^*) \geq 0 \text{ for all } j = 1, \ldots, m, \text{ and} \tag{25}$$

$$\lambda_j^* g_j(x^*) = 0 \text{ for all } j = 1, \ldots, m. \tag{26}$$

Recall that all affine functions are convex, so if $g_j$ is convex for each $i = 1, \ldots, m$, constraint qualification (B) is satisfied; so if that is the case, Theorem 5 tells us that the conditions (24), (25) and (26) are necessary.

The following example, although a bit trivial, illustrates the important point that without Slater's condition, the constraint functions being concave itself (unless they are also convex, so they are affine) does not guarantee that the conditions above are necessary.

**Example 16.** Consider the problem

$$\begin{aligned} \max_{x \in \mathbb{R}} \quad & x \\ \text{s.t.} \quad & -x^2 \geq 0. \end{aligned}$$

Clearly, $g(x) = -x^2$ is a concave function; and the only point that satisfies the constraint is $x = 0$, so it must be, trivially, the solution of the problem.

Form the Lagrangian,

$$\mathcal{L} = x - \lambda x^2,$$

and conditions (24), (25) and (26) for this Lagrangian are

$$\begin{aligned} 1 - 2\lambda x &= 0, \\ -x^2 &\geq 0, \\ -\lambda x^2 &= 0. \end{aligned}$$

However, the solution $x = 0$ does not satisfy the first condition for any $\lambda \geq 0$.

## 6.2 Sufficiency

We showed in Section 3.4 that the first-order conditions are sufficient for a global maximum if the objective function is concave. For maximization problem with inequality constraints, we have a similar result, but we need to impose some conditions on the constraints.

**Theorem 6** (Sufficiency, concave objective function)**.** *Let $S$ be an open subset of $\mathbb{R}^n$, and let $f : S \longrightarrow \mathbb{R}$ and $g_j : S \longrightarrow \mathbb{R}$, $j = 1, \ldots, m$ be continuously differentiable functions. Suppose that*

- *$f$ is concave*
- *and $g_j$ is concave for each $j = 1, \ldots, m$,*

*then if there exist $x^* \in S$ and $\lambda^* \in \mathbb{R}^m_+$ such that*

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0, \tag{27}$$

$$g_j(x^*) \geq 0 \text{ for all } j = 1, \ldots, m, \tag{28}$$

$$\lambda^*_j g_j(x^*) = 0 \text{ for all } j = 1, \ldots, m, \tag{29}$$

*$x^*$ solves the problem*

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \geq 0 \text{ for } j = 1, \ldots, m. \end{aligned} \tag{30}$$

*Proof.* Form the Lagrangian

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x). \tag{31}$$

For any fixed $\lambda \in \mathbb{R}^n_+$, since it is a nonnegative combination of concave functions, $\mathcal{L}$ is concave in $x$. Then by Proposition 6, because $x^*$ satisfies Equation (27), it must be a maximizer of $\mathcal{L}$ on $S$ for any fixed $\lambda \in \mathbb{R}^n_+$. And by Equation (28), $x^*$ is feasible for problem (30).

Take any $x \in S$ satisfying Equation (28); because $x^*$ maximizes $\mathcal{L}$ for any $\lambda \in \mathbb{R}^n_+$, we have

$$\mathcal{L}(x^*, \lambda^*) = f(x^*) + \sum_{j=1}^{m} \lambda_j^* g_j(x^*) \geq f(x) + \sum_{j=1}^{m} \lambda_j^* g_j(x) = \mathcal{L}(x, \lambda^*).$$

Then by Equation (29),

$$f(x^*) = f(x^*) + \sum_{j=1}^{m} \lambda_j^* g_j(x^*) \geq f(x) + \sum_{j=1}^{m} \lambda_j^* g_j(x) \geq f(x),$$

where the last inequality follows from the facts that $\lambda^* \in \mathbb{R}^m_+$ and $g_j(x) \geq 0$ for all $j$. Thus, $x^*$ solves problem (30). ∎

Recall that, if

- $g_j$ is affine for each $j = 1, \ldots, m$, or
- $g_j$ is concave for each $j = 1, \ldots, m$, and Slater's condition holds, that is, there exists $\hat{x} \in S$ such that $g_j(\hat{x}) > 0$ for all $j = 1, \ldots, m$,

then Theorem 5 implies that conditions (27) to (29) are also necessary for the problem.

**Corollary 1** (Necessity and sufficiency). *Let $S \subseteq \mathbb{R}^n$ and let $f : S \longrightarrow \mathbb{R}$ and $g_j : S \longrightarrow \mathbb{R}$, $j = 1, \ldots, m$ be continuously differentiable functions. Suppose that $f$ is concave, and*

- *either $g_j$ is affine for each $j = 1, \ldots, m$,*
- *or $g_j$ is concave for each $j = 1, \ldots, m$, and Slater's condition holds,*

*then $x^* \in S$ solves the problem*

$$\begin{aligned} \max_{x \in S} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \geq 0 \;\; \text{for} \;\; j = 1, \ldots, m \end{aligned}$$

*if and only if there exists $\lambda^* \in \mathbb{R}^m_+$ such that conditions (27), (28), and (29) hold.*

In fact, Theorem 6 holds under alternative assumptions.

**Proposition 8** (Sufficiency, quasiconcave objective function). *Let $S$ be an open subset of $\mathbb{R}^n$, and let $f : S \longrightarrow \mathbb{R}$ and $g_j : S \longrightarrow \mathbb{R}$, $j = 1, \ldots, m$ be continuously differentiable functions. Suppose that*

- *$f$ is quasiconcave*
- *and $g_j$ is quasiconcave for each $j = 1, \dots, m$.*

*If there exist $x^* \in S$ and $\lambda^* \in \mathbb{R}^m_+$ such that (27), (28) and (29) hold, and it is not the case that $\frac{\partial f}{\partial x_i}(x^*) = 0$ for $i = 1, \dots, n$, then $x^*$ solves problem (30).*

The proof of Proposition 8 is beyond the scope of this class, and hence omitted. Interested reader are directed to Section 7.3 of Osborne (2016).

**Example 17.** Consider the problem

$$\begin{aligned} \max_{x \in \mathbb{R}} \quad & -(x-2)^2 \\ \text{s.t.} \quad & x \geq 3. \end{aligned}$$

Since the objective function is concave and the constraint function is affine, so by Corollary 1, the set of solutions of the problem is the set of solutions of conditions (27), (28) and (29).

Form the Lagrangian,

$$\mathcal{L} = -(x-2)^2 + \lambda(x-3),$$

and the necessary and sufficient conditions are

$$\begin{aligned} -2(x-2) + \lambda &= 0, \\ x - 3 \geq 0, \quad \lambda &\geq 0, \\ \lambda(x-3) &= 0. \end{aligned}$$

The complementary slackness condition implies that, either $\lambda \geq 0$ and $x = 3$, or $x \geq 3$ and $\lambda = 0$. If $\lambda = 0$, then the first condition above implies that $x = 2$, contradicts the second condition. If $x = 3$, we have $\lambda = 2$, and all three conditions above are satisfied. Thus, the unique solution for the problem is $x = 3$.

You might feel that Example 17 is too simple, but it illustrates a general procedure for finding solutions of the necessary conditions: We look at the complementary slackness conditions first, which imply that either a Lagrangian multiplier is zero or a constraint is binding; then follow through the implications of each case (in Example 17 there are two cases: (1) $\lambda \geq 0$ and $x = 3$, and (2) $x \geq 3$ and $\lambda = 0$), using the other equations to determine whether it yields a solution or not.

In many cases, one might know some candidates for solutions to a problem. You can try to prove your guess by finding corresponding multipliers that satisfy the sufficient conditions given in the two corollaries below. Importantly, these results do not rely on differentiability of objective function and constraint functions.

**Corollary 2.** *Let $S$ be an open subset of $\mathbb{R}^n$, and let $f : S \longrightarrow \mathbb{R}$ and $g_j : S \longrightarrow \mathbb{R}$, $j = 1, \dots, m$. If there exist $x^* \in S$ and $\lambda^* \in \mathbb{R}^m_+$ such that*

$$\begin{aligned} \mathcal{L}(x^*, \lambda^*) &\geq \mathcal{L}(x, \lambda^*) \text{ for all } x \text{ feasible for problem (30)}, \\ g_j(x^*) &\geq 0 \text{ for all } j = 1, \dots, m, \text{ and} \\ \lambda_j^* g_j(x^*) &= 0 \text{ for all } j = 1, \dots, m, \end{aligned} \tag{32}$$

*then $x^*$ is a solution to problem (30).*

The proof of Corollary 2 is contained in the proof of Theorem 6: there we show that, using differentiability and concavity assumptions, Equation (32) holds; for Corollary 2, we take (32) as given, and derive the same result as in Theorem 6.

**Corollary 3** (Saddlepoint condition)**.** *Let $S$ be an open subset of $\mathbb{R}^n$, and let $f : S \longrightarrow \mathbb{R}$ and $g_j : S \longrightarrow \mathbb{R}$, $j = 1, \dots, m$. Let the Lagrangian be*

$$\mathcal{L}(x, \lambda) = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x).$$

*If there exist $x^* \in S$ and $\lambda^* \in \mathbb{R}^m_+$ such that*

$$\mathcal{L}(x^*, \lambda) \geq \mathcal{L}(x^*, \lambda^*) \geq \mathcal{L}(x, \lambda^*), \tag{33}$$

*then $x^*$ is a solution to problem (30).*

*Proof.* By Corollary 2, it suffices to show that if (33) holds, the three conditions in that corollary hold. Observe that (32) is just "a part of" (33); the rest is left as an exercise. ■

## 6.3 Karush-Kuhn-Tucker (KKT) conditions

In economics and business, we frequently deal with some *nonnegativity constraints*: for example, in most cases a consumer is not allowed to purchase negative amount of goods. In this section, we introduce a convenient approach to deal with problems in which the choice variables are constrained to be nonnegative.

Formally, let $S$ be an open subset of $\mathbb{R}^n$, and let $f$ and $g_j$, $j = 1, \dots, m$, be real-valued continuously differentiable functions defined on $S$ as before. Consider the problem

$$\begin{aligned} \max_{x\in S} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \geq 0 \text{ for } j = 1, \dots, m \\ & x_k \geq 0 \text{ for } k = 1, \dots, n \end{aligned} \tag{34}$$

it is not difficult to see that problem (34) is a special case of the class of problems we dealt with earlier in this section: we can write problem (34) as

$$\begin{aligned} \max_{x\in S} \quad & f(x) \\ \text{subject to} \quad & g_j(x) \geq 0 \text{ for } j = 1, \dots, m, m+1, \dots, m+n \end{aligned}$$

where $g_{m+k}(x) = x_k$ for each $k = 1, \dots, n$. Form the Lagrangian as before:

$$\mathcal{L} = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x) + \sum_{k=1}^{n} \nu_k g_{m+k}(x);$$

assuming that some constraint qualifications (say, one of (A), (B) and (C)) are satisfied, by Theorem 5, the necessary conditions are

$$\nabla_x \mathcal{L}(x^*, \lambda^*) = 0, \tag{35}$$
$$\lambda_j^* \geq 0 \ \text{ for all } j = 1, \dots, m, \tag{36}$$
$$\nu_k^* \geq 0 \ \text{ for all } k = 1, \dots, n, \tag{37}$$
$$g_j(x^*) \geq 0 \ \text{ for all } j = 1, \dots, m+n, \tag{38}$$
$$\lambda_j^* g_j(x^*) = 0 \ \text{ for all } j = 1, \dots, m, \quad \text{and} \tag{39}$$
$$\nu_k^* g_{m+k}(x^*) = 0 \ \text{ for all } k = 1, \dots, n. \tag{40}$$

Now consider the Karush-Kuhn-Tucker (KKT) Lagrangian, which drops the third term (the term related to nonnegative constraints) in the original Lagrangian:[6]

$$\mathcal{L}_{KKT} = f(x) + \sum_{j=1}^{m} \lambda_j g_j(x). \tag{41}$$

[6] To distinguish the KKT Lagrangian from the original one, we use $\mathcal{L}_{KKT}$ to denote the former. When we use it, however, we usually abuse notation and still denote it by $\mathcal{L}$.

Then Equation (43) implies that for each $k = 1, \dots, n$, [7]

$$\frac{\partial \mathcal{L}}{\partial x_k}(x^*) = \frac{\partial \mathcal{L}_{KKT}}{\partial x_k}(x^*) + v_k^*;$$

equivalently,

$$\frac{\mathcal{L}_{KKT}}{\partial x_k}(x^*) = -v_k^* \tag{42}$$

for each $k$. Now (37), (40) and (42) together imply that

$$\frac{\partial \mathcal{L}_{KKT}}{\partial x_k}(x^*) \leq 0 \quad \text{and} \quad x_k^* \frac{\partial \mathcal{L}_{KKT}}{\partial x_k}(x^*) = 0.$$

And for each $j = 1, \dots, m$,

$$\frac{\partial \mathcal{L}_{KKT}}{\partial \lambda_j}(x^*) = g_j(x^*) \geq 0,$$

where the inequality follows from (38); and the equality above implies that Equation (39) can be written as

$$\lambda_j^* \frac{\partial \mathcal{L}_{KKT}}{\partial \lambda_j}(x^*) = 0.$$

Thus, conditions (35) to (40) together imply the following conditions for the KKT Lagrangian:

$$\frac{\partial \mathcal{L}_{KKT}}{\partial x_k}(x^*) \leq 0, \qquad x_k^* \frac{\partial \mathcal{L}_{KKT}}{\partial x_k}(x^*) = 0, \qquad x^* \geq 0, \tag{43}$$

$$\frac{\partial \mathcal{L}_{KKT}}{\partial \lambda_j}(x^*) \geq 0, \qquad \lambda_j^* \frac{\partial \mathcal{L}_{KKT}}{\partial \lambda_j}(x^*) = 0, \qquad \lambda^* \geq 0 \tag{44}$$

for each $k = 1, \dots, n$ and $j = 1, \dots, m$. We call conditions (43) and (44) the **Karush-Kuhn-Tucker (KKT) conditions**.

And if $x^* \in S$ and $\lambda^* \in \mathbb{R}_+^m$ satisfy the KKT conditions, we can let

$$v_k^* = -\frac{\mathcal{L}_{KKT}}{\partial x_k}(x^*),$$

then by (43), we have $v_k^* \geq 0$ and $v_k^* x_k^* = 0$ for all $k = 1, \dots, n$, so we get (35) to (40) back.

Therefore, we conclude that if $(x^*, \lambda^*, v^*)$ satisfies conditions (35) to (40), then $(x^*, \lambda^*)$ satisfies the KKT conditions, and if $(x^*, \lambda^*)$ satisfies the KKT conditions, then we can find $v^* \in \mathbb{R}_+^n$ such that $(x^*, \lambda^*, v^*)$ satisfies conditions (35) to (40). As a consequence, in optimization problems with nonnegativity constraints, whenever conditions (35) to (40) can be used to solve the problem, we may alternatively use the KKT conditions. In other words, whenever conditions (35) to (40) are necessary, so are the KKT conditions; and whenever conditions (35) to (40) are sufficient, the KKT

[7] Again, we suppress variables $\lambda$ and $v$ of $\mathcal{L}$ to save notation.

conditions are sufficient as well.

**Example 18.** Let $u(x_1, x_2)$ be the utility function of a consumer, and $p_i > 0$ is the price of good $i$, $w > 0$ is the consumer's wealth or income. The consumer's consumption choice problem is

$$\begin{aligned} \max_{(x_1,x_2)\in\mathbb{R}^2} \quad & u(x_1, x_2) \\ \text{s.t.} \quad & p_1x_1 + p_2x_2 \le w \\ & x_k \ge 0 \text{ for } k = 1, 2. \end{aligned}$$

The KKT Lagrangian for this problem is

$$\mathcal{L} = u(x_1, x_2) + \lambda (w - p_1x_1 - p_2x_2),$$

and the KKT conditions are

$$\begin{aligned} &\frac{\partial u}{\partial x_1}(x^*) - \lambda^* p_1 \le 0, && x_1^* \left( \frac{\partial u}{\partial x_1}(x^*) - \lambda^* p_1 \right) = 0, && x_1^* \ge 0, \\ &\frac{\partial u}{\partial x_2}(x^*) - \lambda^* p_2 \le 0, && x_2^* \left( \frac{\partial u}{\partial x_2}(x^*) - \lambda^* p_2 \right) = 0, && x_2^* \ge 0, \\ &w - p_1x_1^* - p_2x_2^* \ge 0, && \lambda^*(w - p_1x_1^* - p_2x_2^*) = 0, && \lambda^* \ge 0. \end{aligned}$$

**Example 19.** Consider the problem

$$\begin{aligned} \max_{(x_1,x_2)\in\mathbb{R}^2} \quad & x_1x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 6 \\ & x_1 \ge 0, \quad x_2 \ge 0. \end{aligned}$$

Observe that the constraint functions are affine, so the KKT conditions are necessary. Also, the objective function is continuous and the constraint set is compact (draw a picture), so by the extreme value theorem (Theorem 1), the problem has a solution. Thus the solutions of the problem are the solutions of the KKT conditions that yield the highest values for the function. (Why?)

The KKT Lagrangian is

$$\mathcal{L} = x_1x_2 + \lambda(6 - x_1 - x_2),$$

and the KKT conditions are

$$\begin{aligned} x_2^* - \lambda^* \le 0, \qquad & x_1^*(x_2^* - \lambda^*) = 0, \qquad & x_1^* \ge 0, \\ x_1^* - \lambda^* \le 0, \qquad & x_2^*(x_1^* - \lambda^*) = 0, \qquad & x_2^* \ge 0, \\ 6 - x_1^* - x_2^* \ge 0, \qquad & \lambda^*(6 - x_1^* - x_2^*) = 0, \qquad & \lambda^* \ge 0. \end{aligned}$$

If $x_1^* > 0$, since $x_1^*(x_2^* - \lambda^*) = 0$, we must have $x_2^* = \lambda^*$. If $x_2^* = \lambda^* = 0$, then $x_1^* - \lambda^* \le 0$ implies that $x_1^* \le 0$, a contradiction. Thus, $x_2^* > 0$, but then $x_2^*(x_1^* - \lambda^*) = 0$ implies that $x_1^* = \lambda^* = x_2^* > 0$. Because $\lambda^* > 0$, we have $6 - x_1^* - x_2^* = 0$, so $(x_1^*, x_2^*, \lambda^*) = (3, 3, 3)$.

If $x_1^* = 0$, then if $x_2^* > 0$ we have $\lambda^* = 0$ from the second line of the KKT conditions, but then the first line contradicts $x_2^* > 0$. Thus $x_2^* = \lambda^* = 0$ from the third line.

Thus, there are two solutions of the KKT conditions, (3, 3, 3) and (0, 0, 0). Since the value of the objective function at (3, 3) is greater than the value of the objective function at (0, 0), the solution of the problem is (3, 3).

Although we have discussed many sufficiency theorems in Section 6.2, in many applications, the assumptions of the theorems could be difficult to check, or just might not hold. For example, checking concavity/quasiconcavity of an objective function of two variables is already annoying (see the appendix of the real analysis lecture notes for details); it is just a nightmare for functions of more variables.

The analysis in Example 19 suggests an alternative approach. Note that, so long as the objective function and the constraint functions are defined on an open subset of $\mathbb{R}^n$,[8] whenever $x^*$ is a solution to the problem, and some constraint qualifications hold, it must satisfy the conditions therein. Thus, if the objective function is continuous and the constraint set is compact, then by the extreme value theorem (Theorem 1), a solution must exist. Amongst these candidates, we choose the ones that yield the highest values for the function, and these are the solutions of the original problem. In most cases, there are not too many points satisfying the necessary conditions in Theorem 5, so comparing between the candidates should not be too difficult.

[8] Recall that, if $S$ is open, then $S = \text{int} S$; and $\mathbb{R}^n$ itself is open. In most applications, we have $S = \mathbb{R}^n$, so this condition is rarely violated.

# Appendix

## A Some linear algebra

Let $s^1, s^2, \ldots, s^m \in \mathbb{R}^n$, and let

$$S = \{s^1, s^2, \ldots, s^m\}.$$

We say that a vector $y$ can be expressed as a **linear combination** of vectors in $S$ if there exist $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$ such that

$$y = \sum_{j=1}^{m} \alpha_j s^j.$$

The set $S$ of vectors is said to be **linearly independent** if

$$\sum_{j=1}^{m} \alpha_j s^j = 0 \quad \text{implies} \quad \alpha_j = 0 \ \text{ for all } \ j = 1, \ldots, m.$$

And the set $S$ is said to be **linearly dependent** if it is not linearly independent; that is, there exist $\alpha_1, \alpha_2, \ldots, \alpha_m \in \mathbb{R}$, not all zero, such that

$$\sum_{j=1}^{m} \alpha_j s^j = 0.$$

In words, a set of vectors is linearly dependent if at least one of its element can be expressed as a linear combination of some other elements; and a set of vectors is linearly independent if none of its element can be expressed as a linear combination of some other elements.

**Example 20.** Consider sets

$$S = \{(0, 1, 0), (-2, 2, 0)\} \quad \text{and} \quad T = \{(1, 1, 0), (0, -3, 1), (2, 5, -1)\}.$$

Clearly, none of the elements in $S$ can be written as a linear combination of the other, hence it is linearly independent. And for $T$, observe that

$$(2, 5, -1) = 2(1, 1, 0) - (0, -3, 1),$$

hence it is linearly dependent.

For a finite set of vectors $S$, define its **rank** as the maximal number of linearly independent vectors in $S$. Since a matrix can be viewed as a set containing all its columns, for a matrix $A$, its rank is the maximal number of linearly independent columns. We usually denote the rank of a matrix $A$ by $\text{rank}A$.

**Example 21.** Consider matrices

$$A = \begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -3 & 5 \\ 0 & 1 & -1 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 2 \\ 1 & -3 & 4 \\ 0 & 1 & -1 \end{pmatrix}.$$

It is not difficulty to see that $\operatorname{rank} A = 1$: $(-2, 4) = -2(1, -2)$, so the maximal number of linearly independent columns is just one. By the argument in the previous example we know that the rank of $B$ can be no more than two; and since $(0, -3, 1)$ and $(1, 1, 0)$ are linearly independent, $\operatorname{rank} B = 2$. Finally, because all three column vectors of $C$ are linearly independent, we have $\operatorname{rank} C = 3$.

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