ECN594: Math Bootcamp

Logic and Set Theory

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Several parts of these notes draw heavily on Chapter 1-10 of Halmos (1960), Chapter 1 of Munkres (2000), Section A.1 of Ok (2007), as well as the lecture notes written by Ahmet Altınok for the same class. I have also benefited from Chapter 1 and 2 of Corbae, Stinchcombe, and Zeman (2009) and Chapter 1 of de la Fuente (2000).

Contents

1 Sets: A first look

As Halmos (1960) points out, "the mathematical concept of a set can be used as the foundation of all known mathematics." We adopt a "naïve" point of view of set theory, in the sense that we do not explicitly define a set; rather, we assume that the readers have an intuitive understanding, could be erroneous, of sets.[1] A rigorous treatment along this line, like Halmos (1960), delineates some of the many things that one can correctly do with them by imposing axioms.[2] In these notes, we usually do not explicitly mention these axiom, and you will learn how to deal with sets in an "apprentice" fashion, by observing how we handle them and by working with them yourself.

Examples of sets include a bunch of bananas, a bag of gummy bears, or a collection of numbers. We call members of a set **elements**: an element of a set could be a banana, a gummy bear, or a number. A set itself could also be an element of another set—we will see an example of that soon. If x is an element of A, we write $x \in A$; if x is not an element of A, we express this by writing $x \notin A$. A set could contain finitely or infinitely many elements; later we will define a set that contains no element. In particular, if a set contains exactly one element, we call it a **singleton**.

We say that two sets are **equal** if they have the same elements; if sets A and B are equal, we write $A = B$; otherwise we write $A \neq B$. For example, $\{1, 2\} = \{2, 1\}$, $\{1, 1\} = \{1\}$, and $\{\{1\}\} \neq \{1\}$. (Why?) And if every element of set A is an element of set B, we say that A is a **subset** of B, and we write $A \subseteq B$ (Or sometimes $B \supseteq A$). Observe that if $A \subseteq B$ and $B \subseteq A$, then A and B have the same elements and hence $A = B$. If $A \subseteq B$, but $A \neq B$, then we say that A is a **proper subset** of B, and denote by $A \subsetneq B$.

We pause our discussion on sets for a moment and take a detour to make sure that we understand some basic logic, which will be very helpful for further study in set theory.

2 Logic

A **statement** is an assertion proposing an idea that can be true or false (T or F). For instance, *"all pigeons are black"*, or *"zero is smaller than any positive number"* are statements. In this section we denote a statement by a capital letter; for example, P can denote the statement *"zero is smaller than any positive number"*.

Statements allow us to specify new sets from old ones: each statement about the elements of a set specifies a subset—the subset of those element about which the statement is true.

[1] We can, like in Ok (2007), define a set to be a collection of objects, but "collection" is a synonym of "set", and "object" is left undefined; hence even with this definition, we still require some intuitive understanding, so it does not help us much.

[2] If you are not sure what does "axiom" mean, find its definition at the last sentence in the first paragraph of Section 2.5; I apologize for its appearance before defining it.

Example 1. If A is the set of all men, then

$$B = \{x \in A \,:\, x \text{ is married}\}$$

is the set of all married men; clearly, $B \subseteq A$. Similarly,

$$C = \{x \in A \,:\, x \text{ is not married}\}$$

is the set of all bachelors; again, $C \subseteq A$.

Formally, for every set S and every statement $P(x)$ which stands for "x satisfies statement P",

$$\{x \in S \,:\, P(x)\}$$

is a well-defined subset of S.

Rigorous study in economics and business are based on logical reasoning to prove the validity of a conclusion, Q, from well-defined premises, P. A proof, loosely speaking, is constructed by applying a fixed set of rules to establish that the statement Q is true whenever P is. In the remainder of this section we discuss some rules that allow us to form more and more complex statements, and introduce major types of proofs we frequently use in the remainder of this class.

2.1 And/Or/Not and truth tables

The simplest three ways of constructing new statements from others are using the logical operators "*and*", "*or*" and "*not*". "And" is represented by $\wedge$, which is usually called the **conjunction operator**; a statement $P \wedge Q$ is true if both P and Q are true. "Or" is represented by $\vee$, usually called the **disjunction operator**; and the statement $P \vee Q$ is true if at least one of P and Q is true. And we write $\neg$ for "not", usually known as the **negation operator**; the statement $\neg P$ is true means that P is false. A truth table gives all possible values (T or F) of a statement which is constructed from some simpler statements. The truth table for $P \wedge Q$, $P \vee Q$ and $\neg P$ is shown below.

P	Q	$P \wedge Q$	$P \vee Q$	$\neg P$
T	T	T	T	F
T	F	F	T	F
F	T	F	T	T
F	F	F	F	T

Table 1: The truth table for $P \wedge Q$, $P \vee Q$ and $\neg P$.

The first two columns of Table 1 give possible truth values for the simple statements P and Q. The last three columns give the truth values for $P \wedge Q$, $P \vee Q$ and $\neg P$ conditional on the truth values of P and Q.

Example 2. Consider the statement $\neg[(\neg P) \vee (\neg Q)]$. We evaluate the term inside the square brackets first and then evaluate its negation. The following table gives the truth values for this statement.

P	Q	$\neg P$	$\neg Q$	$(\neg P) \vee (\neg Q)$	$\neg[(\neg P) \vee (\neg Q)]$
T	T	F	F	F	T
T	F	F	T	T	F
F	T	T	F	T	F
F	F	T	T	T	F

2.2 Tautologies and contradictions

A **tautology** is a statement which is true without assumptions; the statement *"$x \neq y$ or $x = y$"* is a tautology, and another example is $x = x$. A **contradiction** is a statement that cannot be true, or always false. Let P be an arbitrary statement, the statment *"P is true and P is false"* is a contradiction; $x \neq x$ is also a contradiction.

2.3 Conditional statement and equivalence

Another important way of constructing statements is connecting two simple statments by "implies" (or equivalently, "if–then–"). We write "P implies Q" as $P \implies Q$; and we call statements constructed this way **conditional statements**. P is called the **antecedent** and Q is called **consequent**. If $P \implies Q$, we usually say P is a **sufficient condition** for Q, or Q is a **necessary condition** for P; Yet two more ways in which we may write the same statement are Q **if** P, and P **only if** Q.

It is important to note that the statement $P \implies Q$ does not make any claim about whether Q is true if P is not true. In other words, the only way in which the statement $P \implies Q$ can fail to hold is if P is guaranteed to be true but Q turns out to be false; otherwise the statement is correct. For example, the statement *"if it rains, then I carry an umbrella"* is not violated by not carrying an umbrella when there is no rain. We call a conditional statement $P \implies Q$ **vacuously true** if P is a contradiction. Table 2 shows that such a statement is indeed true. One example of such a statement is *"if the White House is in France, then the Eiffel Tower is in the US"*—this conditional statement is always true regardless of the consequent is true or not, because the antecedent cannot be satisfied; in other words, the fact that the antecedent is false prevents using the statement to infer anything about the truth value of the consequent.

Yet another statement that can be formed from the statement $P \Longrightarrow Q$ is the statement $Q \Longrightarrow P$, which is called the **converse** of $P \Longrightarrow Q$. We say that P is **equivalent** to Q, denote by $P \Longleftrightarrow Q$, if both the statement $P \Longrightarrow Q$ and its converse $Q \Longrightarrow P$ are true. The statement $P \Longleftrightarrow Q$ can be also expressed as P is a **necessary and sufficient condition** for Q, or P **if and only if** Q.

Example 3. Let A be a statement, we have $A \Longleftrightarrow [\neg(\neg A)]$.

The truth table of conditional statements and equivalent statements is given by Table 2. We

P	Q	$P \Longrightarrow Q$	$Q \Longrightarrow P$	$P \Longleftrightarrow Q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Table 2: The truth table for $P \Longrightarrow Q$ and $P \Longleftrightarrow Q$.

see from Table 2 that, for two statements to be equivalent, they have to be either both correct or both incorrect.

Given a statement of the form $P \Longrightarrow Q$, its **contrapositive** is defined to be the statement $\neg Q \Longrightarrow \neg P$. For example, the contrapositive of the statement *"if $x > 0$, then $x^2 \neq 0$"* is *"if $x^2 = 0$, then $x \leq 0$"*.

Observation 1. *A conditional statement is equivalent to its contrapositive; that is,*

$$(P \Longrightarrow Q) \quad \Longleftrightarrow \quad (\neg Q \Longrightarrow \neg P).$$

Why does Observation 1 hold? If S and T are two statements, we know from Table 2 that $S \Longleftrightarrow T$ is correct if either both S and T are true or both false. Recall that the only way in which the statement $P \Longrightarrow Q$ can be false is if P is true and Q is false; otherwise it is correct. Similarly, the only way in which the statement $\neg Q \Longrightarrow \neg P$ can fail to be correct is if $\neg Q$ is true and $\neg P$ is false; but this is the same as saying that Q is false and P is true. And this, in turn, is precisely the situation in which $P \Longrightarrow Q$ is incorrect. Thus, we see that $P \Longrightarrow Q$ and $\neg Q \Longrightarrow \neg P$ are either both correct or both incorrect, which implies that they are actually equivalent.

2.4 Logical quantifiers

We frequently use two logical quantifiers to construct statements: $\exists$, reads "there exists" (sometimes we also say "for some"), and $\forall$, reads "for all". For example, $(\exists x \in A)$ $[P(x)]$ states that there exists x in set A such that P holds for x, and $(\forall x \in A)$ $[\neg P(x)]$ says that for all x in set A, P does not hold.

We need to be careful when negating statements containing logical quantifiers. For example, consider the statement *"for every $x \in A$, Q holds."* How do we form the negation of this statement? Let us translate the problem into the language of sets. Let B denote the set of all those elements x for which Q holds; that is,

$$B = \{x : Q(x)\}.$$

Then the statement becomes that *"A is a subset of B"*, and its negation is *"A is not a subset of B"*, which is equivalent to the statement *"there exists at least one element of A that does not belong to B"*. By definition of the set B, this becomes *"there exists (or for some) $x \in A$, Q does not hold"*. Hence,

$$\neg\{(\forall x \in A)\ [Q(x)]\} \iff (\exists x \in A)\ [\neg Q(x)].$$

In a statement we often see multiple quantifiers: for instance, let A and B be sets; then $(\exists x \in A, \forall y \in B)\ [P(x, y)]$ asserts that there exists x in set A, such that for all y in set B, the statement P holds for x and y. (How should we negate this statement?) In such cases, the order of the quantifiers are very important: once the order of quantifiers change, usually the meaning of the statement would change as well. For example, $(\exists x \in A, \forall y \in B)\ [P(x, y)]$ is different from the statement $(\forall x \in A, \exists y \in B)\ [P(x, y)]$, which asserts that for all $x \in A$, we can find some $y \in B$, such that the statement P holds for x and y.

2.5 Proofs

We follow Corbae et al. (2009) to define a few terms as follows. A **theorem** or **proposition** is a statement that we prove to be true. A **lemma** is a result we use to prove another theorem. A **corollary** is a statement whose proof is supposed to follow directly from the previous theorem. A **definition** is a statement that is true by interpreting one of its terms in such a way as to make the statement true. An **axiom** or **assumption** is a statement that is taken to be true without proof.

We now introduce basic proof techniques that we will frequently encounter in economics and business. The most intuitive and familiar one is the **direct proof**: to prove $P \implies Q$, we assume that P holds, and show that Q must be true, possibly throught some intermediate steps (say, $P \implies K \implies R \implies Q$). And to verify the equivalence $P \iff Q$, we can split it into checking $P \implies Q$ and $Q \implies P$.

Example 4. We directly prove the following statement: if x is odd, then x^2 is odd. We suppose that x is odd, and we try to show that so is x^2. Since x is odd, there exists some integer k such that $x = 2k + 1$; hence

$$x^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1.$$

Because k is an integer, so is $2k^2 + 2k$; call it z. Therefore, there exists an integer z such that $x^2 = 2z + 1$, so x^2 is indeed odd. This completes the proof.

Another commonly used proof technique is **proof by contraposition**. By Observation 1, a statement $P \implies Q$ is equivalent to its contrapositive $\neg Q \implies \neg P$; thus, to prove $P \implies Q$, it suffices to show that $\neg Q \implies \neg P$.[3]

Example 5. Let x be an integer, and we prove the following statement by contraposition: if x^2 is even, then x is even. We suppose that x is *not* even, and we try to show that x^2 is not as well. Since x is not even, it must be odd; but the statement we proved in Example 4 shows that x^2 must also be odd, hence it is not even. In fact, this statement is just the contrapositive of the statement in Example 4.

Example 6. Let $\mathbb{R}$ be the set of all real numbers, which is usually called the real line.[4] Let $a, b \in \mathbb{R}$, we prove the following statement: *"$a \leq b + \varepsilon$ for any $\varepsilon > 0$ implies $a \leq b$"*, or more compactly,

$$(\forall \varepsilon > 0)\ [a \leq b + \varepsilon] \implies a \leq b.$$

It suffices to prove its contrapositive:

$$a > b \implies (\exists \varepsilon > 0)\ [a > b + \varepsilon].$$

Because $a > b$ we have $a - b > 0$; so let

$$\varepsilon = \frac{a-b}{2} < a - b,$$

adding b on both sides of the inequality above, we get

$$b + \varepsilon < b + (a - b) = a.$$

[3]For example, consider the statement *"if it is raining today, then I do not go to class"*: it is equivalent to its contrapositive *"if I go to class, then it is not raining today"*. So if we want to prove the first statement, it suffices to prove the second statement. Note that both statements above are **not** equivalent to the statement *"if I do not go to class, then it is raining today"*, which is the converse of the first statement.

[4]In these notes we assume that the readers know some basic properties of $\mathbb{R}$; if you are not sure about this, read Appendix B of Sundaram (1996) or Section 1.6 of de la Fuente (2000). If you would like to know more, Section A.2 of Ok (2007) offers a comprehensive treatment.

Because $a > b$, we have $a - b > 0$, and consequently $\varepsilon = (a - b)/2 > 0$. Therefore, we have found an $\varepsilon > 0$ such that $a > b + \varepsilon$, and we are done.

The third commonly used technique is proof by **contradiction**: we assume the statement to be proved is not true, and then derive a contradiction. This is essentially a special case of proof by contraposition. (Why?)

Example 7. We use proof by contradiction to prove the so-called "Pigeonhole Principle": let m and n be positive integers with $m > n$; suppose m objects are put into n boxes. Then some box contains at least two objects.

Suppose (toward contradiction) that each box contains at most one object. (How did we negate the statement to be proved?) Then the total number of objects is at most

$$1 + 1 + \ldots + 1 = n < m,$$

a contradiction.

When we want to prove a statement $S(n)$ is true for every positive integer, one special proof technique, called **proof by induction**, is particularly useful.[5] To use induction, we prove two things:

- **Base case**: The statement is true in the case where $n = 1$.
- **Inductive step**: If the statement is true for $n = k$, then the statement is also true for $n = k + 1$.

This actually produces an infinite chain of implications:

- The statement is true for $n = 1$
- If the statement is true for $n = 1$, then it is also true for $n = 2$
- If the statement is true for $n = 2$, then it is also true for $n = 3$
- If the statement is true for $n = 3$, then it is also true for $n = 4$
- … …

Together, these implications prove the statement for all positive integer values of n.[6]

[5] Of course, induction is usually not the only way to prove a statement for all positive integers. More importantly, proof by inductions can be applied in many circumstances other than the case of positive integers.

[6] It does not prove the statement for non-integer values of n, or integers strictly less than 1.

Example 8. We use induction to prove that

$$1 + 2 + \ldots + n = \frac{n(n+1)}{2} \tag{1}$$

for all positive n.

- Base case: when $n = 1$, the left-hand side of Equation (1) is one, and the right-hand side is $(1 \cdot 2)/2 = 1$, hence Equation (1) holds.
- Inductive step: suppose Equation (1) holds for $n = k$:

  $$1 + 2 + \ldots + k = \frac{k(k+1)}{2}. \tag{2}$$

  Now consider $n = k + 1$; by Equation (2), we have

  $$1 + 2 + \ldots + k + (k+1) = \frac{k(k+1)}{2} + k + 1 = \left(\frac{k}{2} + 1\right)(k+1) = \frac{(k+1)(k+2)}{2},$$

  hence Equation (1) also holds for $n = k + 1$.

Therefore, Equation (1) is true for all positive integers.

3 Set theory: A brief introduction

Now we are ready to go back to our discussion on set theory. We start from a series of basic but important set operations: we discuss some of their key properties, and point out some connections between set operators and logical operators we studied in Section 2. Then we proceed to a very important concept, called relation, which is a generalization of the (intuitively familiar) concept of a function; the discussion on relations will allow us to study functions in a rigorous way.

Before proceeding, we define two sets that are going to be useful for us. An **empty set** is the set that contains no element; we denote it by $\varnothing$. We can define

$$\varnothing = \{x : x \neq x\};$$

observe that the statement defining this set is a contradiction. The empty set is unique. (Why?)

Claim 1. *$\varnothing \subseteq S$ for every set S.*

To prove the claim, it suffices to show that for every $x \in \varnothing$, $x \in S$ for every set S. But since there does not exist any $x \in S$, the above statement is vacuously true. Therefore, the claim is also true.

In fact, to prove that something is true about the empty set, we can always prove that it cannot be false. Suppose, for example, Claim 1 is false, so there exists a set T such that $\emptyset$ is not a subset of T. This could be the case only if there exists $y \in \emptyset$ such that $y \notin T$. But the empty set contains no elements, a contradiction.

Let S be a set; if $S \neq \emptyset$, we say that S is **nonempty**. For instance, $\{\emptyset\}$ is a nonempty set. Indeed, $\{\emptyset\} \neq \emptyset$:[7] $\{\emptyset\}$ contains one element, which is the the empty set, while $\emptyset$ contains nothing; so the two sets do not contain the same elements, which means that they are not equal.

We define the collection[8] of all subsets of a given set S as

$$\mathscr{P}(S) = \{T \,:\, T \subseteq S\},$$

which is called the **power set** of S.[9] For instance,

$$\mathscr{P}(\{1,2\}) = \{\emptyset, \{1\}, \{2\}, \{1,2\}\};$$

and $\mathscr{P}(\emptyset) = \{\emptyset\}$, $\mathscr{P}(\mathscr{P}(\emptyset)) = \{\emptyset, \{\emptyset\}\}$, and $\mathscr{P}(\mathscr{P}(\mathscr{P}(\emptyset))) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$, and so on. Are the above three sets equal?

3.1 Basic operations

Given two sets A and B, one can form a set from them that consists of all the elements of A together with all the elements of B. This set is called the **union** of A and B and is denoted by $A \cup B$. Formally, we define

$$A \cup B = \{x \,:\, x \in A \text{ or } x \in B\}.$$

Another way to form a set from A and B is to take the common part of them. This set is called the **intersection** of A and B, and is denoted by $A \cap B$. Formally. we define

$$A \cap B = \{x \,:\, x \in A \text{ and } x \in B\}.$$

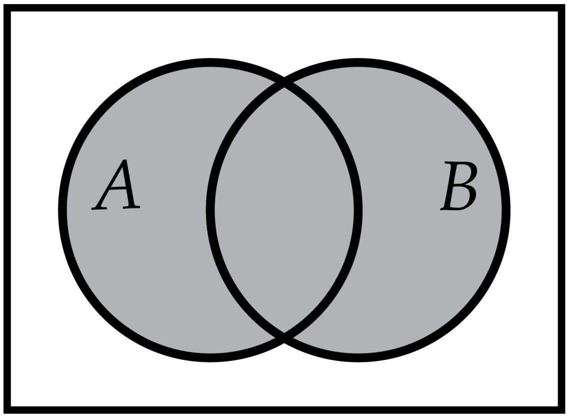
If $A \cap B = \emptyset$, so the two sets have no common elements, we say that A and B are **disjoint**. If $A \subseteq B$, then $A \cup B = B$ and $A \cap B = A$; and in particular, for any set S, since we have $\emptyset \subseteq S$ (Claim 1), $\emptyset \cup S = S$ and $\emptyset \cap S = \emptyset$.

Proposition 1. *Taking unions and and intersections are*

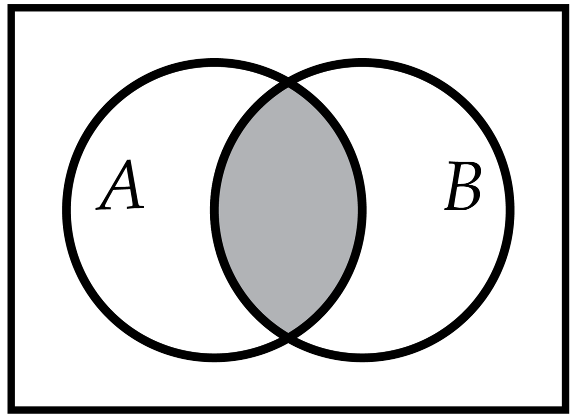
[7] As Ok (2007) notes, an empty box is not the same thing as nothing.

[8] In these notes the term "collection" is used to describe a set of sets. A collection in this sense is a legitimate set, the purpose of using a different term is just to stress that it contains sets as elements.

[9] Some authors prefer to denote the power set of a given set S by 2^S, which is motivated by the fact that if S contains k elements, where k is a finite nonnegative integer, its power set has exactly 2^k elements.



$A \cup B$



$A \cap B$

- *commutative:*

$$A \cap B = B \cap A \quad \textit{and} \quad A \cup B = B \cup A$$

  for any sets A and B;

- *associative:*

$$A \cap (B \cap C) = (A \cap B) \cap C \quad \textit{and} \quad A \cup (B \cup C) = (A \cup B) \cup C$$

  for any sets A, B and C; and

- *distributive:*

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \textit{and} \quad A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

  for any sets A, B and C.

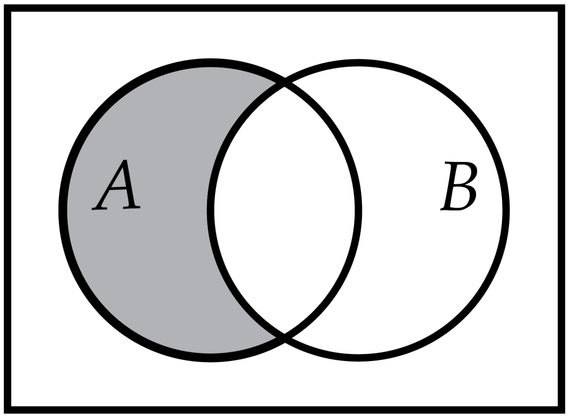
Proof. Exercise. ∎

The **difference** between two sets A and B is defined as the set consisting of those elements of A that are not in B, and is denoted by $A \backslash B$. Formally,

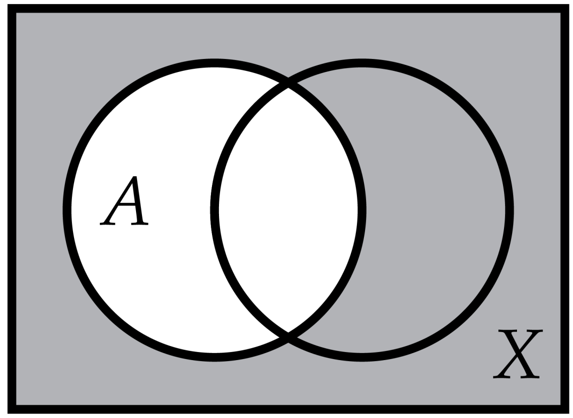
$$A \backslash B = \{x : x \in A \text{ and } x \notin B\}.$$

To make sure that you understand the definition, it is helpful to check the following facts by yourself: $S \backslash \emptyset = S$, $S \backslash S = \emptyset$, and $\emptyset \backslash S = \emptyset$ for any set S.

If a universal set X of which all sets we concern are subsets is well understood, we can define the **complement** of A, which contains the elements that are (in X but) not in A, and is denoted by A^c; that is, $A^c = X \backslash A$. It is straightforward to check that $A \backslash B = A \cap B^c$, and $X^c = \emptyset$, $\emptyset^c = X$.



$A \backslash B$



$A^c = X \backslash A$

Proposition 2 (De Morgan's laws). *If A, B and C are sets, then*

(1) $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$, *and*

(2) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

In particular, taking $A = X$, we have $(B \cup C)^c = B^c \cap C^c$ *and* $(B \cap C)^c = B^c \cup C^c$.

Proof. We prove (1) only; (2) is left as an exercise. Suppose $x \in A \backslash (B \cup C)$, so $x \in A$ and $x \notin B \cup C$. By saying that $x \notin B \cup C$, we mean $x \notin B$ and $x \notin C$; so we have $x \in A$ and $x \notin B$ and $x \notin C$. But this is the same as ($x \in A$ and $x \notin B$) and ($x \in A$ and $x \notin C$); hence, $x \in (A \backslash B) \cap (A \backslash C)$. Thus, we have $A \backslash (B \cup C) \subseteq (A \backslash B) \cap (A \backslash C)$.

For the other direction, suppose that $x \in (A \backslash B) \cap (A \backslash C)$, so $x \in (A \backslash B)$ and $x \in (A \backslash C)$. Thus $x \in A$ and ($x \notin B$ and $x \notin C$), which implies $x \in A$ and $x \notin (B \cup C)$. But this is just $x \in A \backslash (B \cup C)$. Hence, $(A \backslash B) \cap (A \backslash C) \subseteq A \backslash (B \cup C)$. Thus we conclude that $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$. ■

Recall that we use the term "collection" to refer to a set whose elements are sets;[10] to avoid discussing uninteresting cases, when we say "collection" we mean a nonempty one. We usually denote collections by script letters, say $\mathcal{C}$; and we denote an element of a collection, which is also a set, by capital letters, say C. The union of all members of collection $\mathcal{C}$, denoted as $\cup \mathcal{C}$, or $\bigcup \{C : C \in \mathcal{C}\}$, or $\bigcup_{C \in \mathcal{C}} C$, is defined as the set

$$\{x : x \in C \text{ for some } C \in \mathcal{C}\}.$$

Similarly, the intersection of all sets in $\mathcal{C}$, denoted as $\bigcap \mathcal{C}$, or $\cap \{C : C \in \mathcal{C}\}$, or $\cap_{C \in \mathcal{C}} C$, is defined as the set

$$\{x : x \in C \text{ for each } C \in \mathcal{C}\}.$$

[10] Some authors prefer to use the term "family".

A common way of specifying a collection $\mathcal{C}$ of sets is by designating a set I as a set of indices, and by defining

$$\mathcal{C} = \{C_i \,:\, i \in I\}\,.$$

In this case, $\cup\mathcal{C}$ may be denoted as $\bigcup_{i\in I} C_i$. If $I = \{k, k+1, \dots, K\}$ for some integers k and K with $k < K$, then we often write $\bigcup_{i=k}^{K} C_i$. Similarly, if $I = \{k, k+1, \dots\}$ for some integer k, then we may write $\bigcup_{i=k}^{\infty} C_i$. And $\cap\mathcal{C}$ may be denoted as $\bigcup_{i\in I} C_i$, and so on.

Proposition 3 generalizes ditributivity of unions and intersections, as well as the De Morgan's Laws, to an arbitrary collection of sets. Its proof is analogous to the proof of Proposition 2, hence omitted.

Proposition 3. *Let A be a set and $\mathcal{B}$ a collection of sets. Then*

$$A \cap \bigcup\mathcal{B} = \bigcup\{A \cap B \,:\, B \in \mathcal{B}\} \quad \textit{and} \quad A \cup \cap\mathcal{B} = \cap\{A \cup B \,:\, B \in \mathcal{B}\}.$$

Moreover,

$$A\backslash \cup \mathcal{B} = \cap\{A\backslash B \,:\, B \in \mathcal{B}\} \quad \textit{and} \quad A\backslash \cap \mathcal{B} = \bigcup\{A\backslash B \,:\, B \in \mathcal{B}\};$$

and if X is a universal set, putting $A = X$ in the above equation yields

$$(\cup\mathcal{B})^c = \cap\{B^c \,:\, B \in \mathcal{B}\} \quad \textit{and} \quad (\cap\mathcal{B})^c = \bigcup\{B^c \,:\, B \in \mathcal{B}\}.$$

The set operations we introduced above are, unsurprisingly, tightly related to logical operators.

Proposition 4. *Let sets A and B be defined as*[11]

$$A = \{x \,:\, P(x)\}, \quad \textit{and} \quad B = \{x \,:\, Q(x)\},$$

respectively. Then[12]

$$A \cup B = \{x \,:\, P(x) \vee Q(x)\}, \;\; A \cap B = \{x \,:\, P(x) \wedge Q(x)\}, \;\; \textit{and} \;\; A^c = \{x \,:\, \neg P(x)\}.$$

Moreover, $A \subseteq B$ if and only if $P \implies Q$ is true, and $A = B$ if and only if $P \iff Q$ is true. $P \implies Q$ is vacuously true if $A = \emptyset$.

Proof. We argue that $A \cup B = \{x \,:\, P(x) \vee Q(x)\}$ first. If $x \in A \cup B$, we have $x \in A$ or $x \in B$. By definition of A and B, we know that either P holds for x, or Q is true for x (or both). Therefore,

[11] Recall that $P(x)$ is the statement *"statement P holds for x"*.
[12] $A \cup B = \{x \,:\, P(x) \vee Q(x)\}$ can be equivalently expressed as $x \in A \cup B$ if and only if $P(x) \vee Q(x)$.

we must have $P(x) \vee Q(x)$; so $A \cup B \subseteq \{x : P(x) \vee Q(x)\}$. If $y \in \{x : P(x) \vee Q(x)\}$, at least one of P and Q holds for y, so either $y \in A$ or $y \in B$. Consequently, $y \in A \cup B$, which implies that $\{x : P(x) \vee Q(x)\} \subseteq A \cup B$. Thus, $A \cup B = \{x : P(x) \vee Q(x)\}$. The proof of the other two equalities is left as an exercise.

Now we show that $A \subseteq B$ if and only if $P \implies Q$ is true. Suppose $A \subseteq B$, so $x \in A$ implies $x \in B$. By definition of A and B, we know that for each x, if P holds, Q must also be true; but this is exactly $P(x) \implies Q(x)$. Now suppose $P \implies Q$ is true, so if $x \in A$, we must have $x \in B$; so $A \subseteq B$. The other two statements are direct consequences of this one. ∎

3.2 Relations

An **ordered pair** is an ordered list (a, b) consisting of two objects a and b, where "ordered" means that for any two ordered pairs (a, b) and (c, d), $(a, b) = (c, d)$ if and only if $a = c$ and $b = d$. In particular, (a, b) and (b, a) are regarded as distinct unless $a = b$. The **Cartesian product**, usually just **product**, of two nonempty sets A and B, denoted as $A \times B$, is defined as the set of all ordered pairs (a, b) for which a is an element of A and b is an element of B. Formally,

$$A \times B = \{(a, b) : a \in A \text{ and } b \in B\}.$$

We often write A^2 for $A \times A$.

Example 9. Let $A = \{u, v\}$ and $B = \{x, y, z\}$. Then

$$A \times B = \{(u, x), (u, y), (u, z), (v, x), (v, y), (v, z)\},$$

and

$$A^2 = \{(u, u), (u, v), (v, u), (v, v)\}.$$

Claim 2. *For any sets A, B, and C,*

$$A \times (B \cap C) = (A \times B) \cap (A \times C) \quad \textit{and} \quad A \times (B \cup C) = (A \times B) \cup (A \times C).$$

Proof. Exercise. ∎

The idea of Cartesian product can be substantially generalized: for each positive integer n, define an n-**tuple** as a list $(a_1, \dots, a_n)$ that $(a_1, \dots, a_n) = (a'_1, \dots, a'_n)$ if and only if $a_i = a'_i$ for each $i = 1, \dots, n$. The Cartesian product of n nonempty sets $A_1, \dots, A_n$, is then defined as

$$A_1 \times \cdots \times A_n := \{(a_1, \dots, a_n) : a_i \in A_i, i = 1, \dots, n\}.$$

We often write $\bigtimes_{i=1}^{n} A_i$ to denote $A_1 \times A_2 \times \cdots \times A_n$, and $S^n = S \times S \times \cdots \times S$.

Let X and Y be two nonempty sets. A subset R of $X \times Y$ is called a (binary) **relation from** X **to** Y, or simply a **relation** if X and Y can be inferred from context. If $X = Y$, that is, if R is a relation from X to X, we simply say that it is a relation on X. So R is a relation on X if and only if $R \subseteq X^2$. If $(x, y) \in R$, then we think of R as associating the objects x and y, or the relation R holds for the ordered pair (x, y).

Intuitively, the concept of "relation" provides a way to formalize the idea that two objects stand in a certain relationship to each other. For example, if x and y are real numbers, one may be larger than the other; if x and y denote consumption bundles, a given consumer may prefer one to the other. The notation xRy, interpreted as "x stands in a certain relation to y", can be more suggestive than the notation $(x, y) \in R$. As a consequence, we usually write xRy instead of $(x, y) \in R$ for simplicity.

Example 10. Let $A = \{0, 1, 2, 3, 4\}$. The relations $\leq$, $<$, $=$, and $\neq$ on A^2 can be represented by the four panels in Figure 1, respectively.

4	$\otimes$	$\otimes$	$\otimes$	$\otimes$	$\otimes$
3	$\otimes$	$\otimes$	$\otimes$	$\otimes$	
2	$\otimes$	$\otimes$	$\otimes$		
1	$\otimes$	$\otimes$			
0	$\otimes$				
$y \uparrow / x \rightarrow$	0	1	2	3	4

$\leq$

4	$\otimes$	$\otimes$	$\otimes$	$\otimes$	
3	$\otimes$	$\otimes$	$\otimes$		
2	$\otimes$	$\otimes$			
1	$\otimes$				
0					
$y \uparrow / x \rightarrow$	0	1	2	3	4

$<$

4					$\otimes$
3				$\otimes$	
2			$\otimes$		
1		$\otimes$			
0	$\otimes$				
$y \uparrow / x \rightarrow$	0	1	2	3	4

$=$

4	$\otimes$	$\otimes$	$\otimes$	$\otimes$	
3	$\otimes$	$\otimes$	$\otimes$		$\otimes$
2	$\otimes$	$\otimes$		$\otimes$	$\otimes$
1	$\otimes$		$\otimes$	$\otimes$	$\otimes$
0		$\otimes$	$\otimes$	$\otimes$	$\otimes$
$y \uparrow / x \rightarrow$	0	1	2	3	4

$\neq$

Figure 1: Relations $\leq$, $<$, $=$, and $\neq$ on $\{0, 1, 2, 3, 4\}^2$.

Example 11. Let

$$X = \{\text{Austin, Chicago, Phoenix, Sacramento}\}$$

and

$$Y = \{\text{Arizona, California, Illinois, Texas}\}.$$

Then the relation

$$R = \{(\text{Austin, Texas}), (\text{Phoenix, Arizona}), (\text{Sacramento, California})\} \subseteq X \times Y$$

expresses "is the state capital of".

3.3 Functions

When we talk about a function, we mean a rule that transforms the objects in a given set to those of another. We can now use the notion of a binary relation to formalize the idea. Let X and Y be nonempty sets. A **function** f that maps X into Y, denoted as $f : X \longrightarrow Y$, is a relation $f \subseteq X \times Y$ such that

(i) for every $x \in X$, there exists a $y \in Y$ such that xfy;

(ii) for every $y, z \in Y$ with xfy and xfz, we have $y = z$.

In words, f is a relation that *for each* x in X there is *a unique* element y in Y such that xfy. From now on, if f is a function, we shall adopt the familiar notation $f(x) = y$ instead of xfy.[13]

For any $f : X \longrightarrow Y$, X is called the **domain** of f and Y the **codomain** of f. For any $A \subseteq X$, we define the **image** of A under f by

$$f(A) = \{y \in Y : f(x) = y \text{ for some } x \in A\}.$$

In particular, we call $f(X)$ the **range** of f; in words, it is the set of elements y in Y for which there exists an $x \in X$ such that $f(x) = y$. And we say that two functions f and g are **equal**, denote by $f = g$, if they have the same domain and codomain, and $f(x) = g(x)$ for all x in their common domain.

Example 12. All relations in Example 10 and Example 11 except "=" are not functions. Which of (i) and (ii) do they fail to satisfy?

Let $f : X \longrightarrow Y$ be a function. If the range coincides with its codomain, that is, if $f(X) = Y$, then one refers to it as a **surjective function** (or simply a **surjection**).[14] Equivalently, f is surjective if for every element y in the codomain Y, there is at least one element x in the domain X such that $f(x) = y$. Note that we do not require such x to be unique—the function f may map

[13] This familiar notation does not cause any problem because (i) and (ii) hold. (Why?) For those who still find difficult to find the connection between our formal definition of a function and the intuitive "rule of transformation" definition, note that the "behavior" of "a rule" f is completely identified by the set $\{(x, f(x)) \in X \times Y : x \in X\}$, which is actually the formal definition of f.

[14] Some authors prefer to say that f maps X **onto** Y (or just f is onto).

one or more elements of X to the same element of Y. If f maps distinct elements in its domain to distinct elements in its codomain, that is, if $x \neq y$ implies $f(x) \neq f(y)$ for all $x, y \in X$, then we say that f is an **injective function** (or simply **injection**).[15] Finally, if f is both injective and surjective, then it is called a **bijective function** (or a **bijection**). Figure 2 illustrates surjective, injective and bijective functions.

Example 13. Let $A = \{0, 1, 2, 3, 4\}$. The bijective functions $f(x) = x$ and $g(x) = 4 - x$ on A can be represented by

4					$\otimes$
3				$\otimes$	
2			$\otimes$		
1		$\otimes$			
0	$\otimes$				
$y \uparrow / x \rightarrow$	0	1	2	3	4

$f(x) = x$

4	$\otimes$				
3		$\otimes$			
2			$\otimes$		
1				$\otimes$	
0					$\otimes$
$y \uparrow / x \rightarrow$	0	1	2	3	4

$g(x) = 4 - x$

Example 14. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^2$. Clearly f is not surjective: observe that $-1 \in \mathbb{R}$, but $-1 \notin f(\mathbb{R})$, so $f(\mathbb{R})$ is a proper subset of $\mathbb{R}$. We claim that f is not injective. To see this, note that the definition of a injective function is equivalent to (why?)

$$f(x) = f(y) \text{ implies } x = y \text{ for all } x, y \in \mathbb{R}, \tag{3}$$

so it suffices to check (3). But we have $f(2) = f(-2) = 4$, so Equation (3) fails to hold, which implies that f is not injective.

Now let $g : \mathbb{R} \longrightarrow \mathbb{R}_+$ be defined by $g(x) = x^2$, where

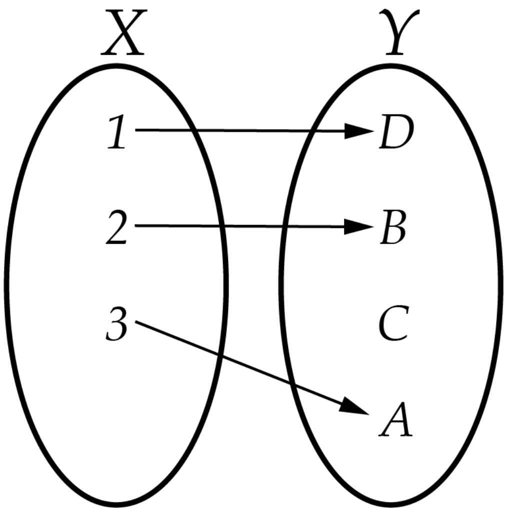
$$\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}.$$

g is surjective since $g(\mathbb{R}) = \mathbb{R}_+$; but it is not injective. Finally, let $h : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined by $h(x) = x^2$, where $\mathbb{R}_+$ is the set of all positive integers. h is bijective: we have $f(\mathbb{R}_+) = \mathbb{R}_+$, and for every $x, y \in \mathbb{R}_+$, $f(x) = f(y)$ must imply $x = y$ (because negative numbers are "excluded").

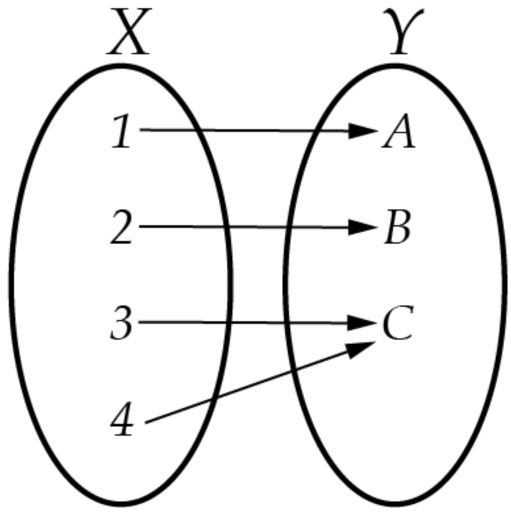
The insights of Example 14 can be substantially generalized.

Observation 2. *Let $f : X \longrightarrow Y$ be an injective function. Then $g : X \longrightarrow f(X)$ is bijective.*

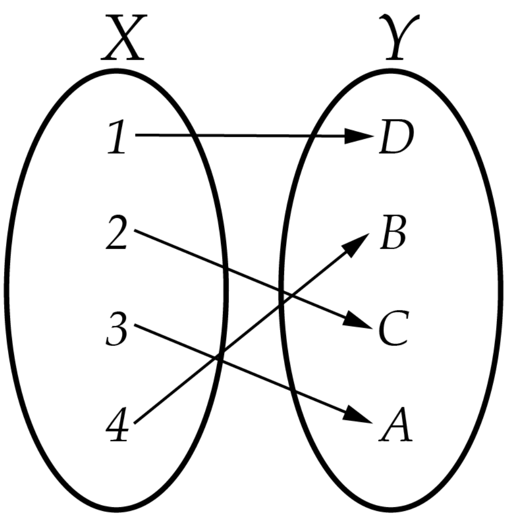
[15]Some authors prefer to say that f is **one-to-one**.



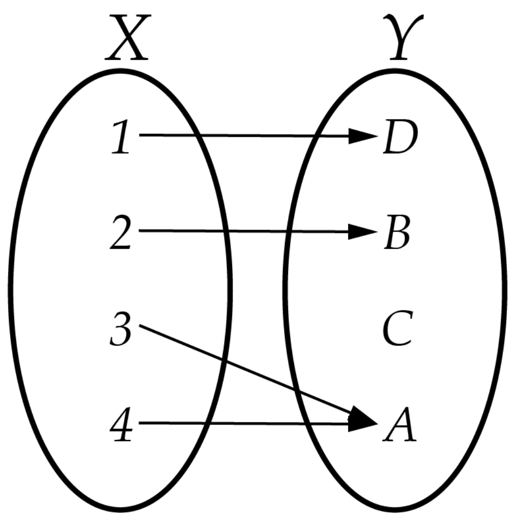
An injective function



A surjective function



An bijective function



A function that is neither injective nor surjective

Figure 2: Some examples of functions

The **inverse image** of a set B in Y, denoted as $f^{-1}(B)$, is defined as the set of all x in X that f maps to B, that is,

$$f^{-1}(B) := \{x \in X : f(x) \in B\}.$$

Observe that f is surjective if and only if the inverse image under f of each nonempty subset of Y is a nonempty subset of X, and f is injective if and only if the inverse image of each singleton in $f(X)$ be a singleton in X. (Why?)

The following proposition characterizes the relationship between images and inverse images.

Proposition 5. *Let $f : X \longrightarrow Y$ be a function.*

(1) If $B \subseteq Y$, then $f\left(f^{-1}(B)\right) \subseteq B$; if f is surjective, $f\left(f^{-1}(B)\right) = B$.

(2) If $A \subseteq X$, then $A \subseteq f^{-1}(f(A))$; if f is injective, $A = f^{-1}(f(A))$.

Proof. Exercise. ∎

Sometimes we are interested in whether we can express the image (or the inverse image) of a union/intersection of a collection as the union/intersection of the images (inverse images) of each set in the collection. The following result says that inverse images are more "well-behaved" than images in that regard.

Proposition 6. *Let X and Y be nonempty sets, and $f : X \longrightarrow Y$ is a function. For any collection $\mathcal{A} \subseteq \mathcal{P}(X)$ and $\mathcal{B} \subseteq \mathcal{P}(Y)$ we have*

$$f(\cup\mathcal{A}) = \bigcup\{f(A) : A \in \mathcal{A}\} \quad \textit{and} \quad f(\cap\mathcal{A}) \subseteq \cap\{f(A) : A \in \mathcal{A}\},$$

whereas

$$f^{-1}(\cup\mathcal{B}) = \bigcup\left\{f^{-1}(B) : B \in \mathcal{B}\right\} \quad \textit{and} \quad f^{-1}(\cap\mathcal{B}) = \bigcap\left\{f^{-1}(B) : B \in \mathcal{B}\right\}.$$

Proof. Exercise. ∎

In fact, let $f : X \longrightarrow Y$, $f(A \cap B) \supseteq f(A) \cap f(B)$ for all $A, B \subseteq X$ if and only if f is injective. The "if" part follows from Proposition 6; for the "only if" part, we argue by contraposition: if f is not injective, there exists $u, v \in X$ such that $u \neq v$ but $f(u) = f(v) = y$; but then

$$f(\{u\} \cap \{v\}) = f(\varnothing) = \varnothing \subsetneq \{y\} = f(\{u\}) \cap f(\{v\}),$$

which completes the proof since the antecedent is also false.

For any function $f : X \longrightarrow Y$, define the relation from Y to X[16]

$$f^{-1} = \{(y, x) \in Y \times X : f(x) = y\}.$$

This relation "reverses" f in the sense that if x is mapped to y by f, then f^{-1} maps y "back" to x. If f^{-1} is a function, we say that f is **invertible** and f^{-1} is the **inverse** of f.

Example 15. Let $f : \mathbb{R} \longrightarrow \mathbb{R}_+$ be defined by $f(x) = x^2$. f is not invertible because for $y = 1$, we have $f(1) = f(-1) = y$, so (ii) in the definition of a function is violated. But for $g : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ be defined by $f(x) = x^2$, it is invertible with $g^{-1}(y) = \sqrt{y}$ for all $y \in \mathbb{R}_+$.

Comparing Example 14 with Example 15, one may suspect that invertibility of a function is closely related to whether it is bijective. In fact, these two conditions are equivalent.

Proposition 7. *Let X and Y be two nonempty sets. A function $f : X \longrightarrow Y$ is invertible if and only if it is a bijection.*

Proof. We prove the "if" direction first. Suppose f is a bijection, so it is both injective and surjective. To show that f is invertible, it suffices to show that the relation

$$f^{-1} = \{(y, x) \in Y \times X : f(x) = y\}$$

is indeed a function; that is, it satisfies

(i) for each $y \in Y$, there exists $x \in X$ such that $f(x) = y$, and

(ii) for every $u, v \in X$ such that $f(u) = f(v) = y$ for some $y \in Y$, we have $u = v$.

Since f is surjective, for every $y \in Y$, there exists $x \in X$ such that $f(x) = y$, and this is exactly (i). For (ii), suppose $f(u) = f(v) = y$ for some $y \in Y$. Because f is injective, $f(u) = f(v)$ implies $u = v$, which establishes (ii). Thus, f^{-1} is indeed a function; by definition, f is an invertible function.

For the "only if" direction, suppose f is invertible, so (i) and (ii) must hold. But these two conditions essentially coincide with the definition of f being surjective and injective, respectively; hence, f is bijective. ∎

The following corollary is an immediate consequence of Proposition 7 and Observation 2.

Corollary 1. *Let $f : X \longrightarrow Y$ be an injective function. Then $g : X \longrightarrow f(X)$ is invertible.*

[16] Recall that, since (y, x) is an ordered pair, $(y, x) \neq (x, y)$!

If f is a function that maps X into Y and g is a function maps Y into Z, then every element in the range of f belongs to the domain of g, and consequently $g(f(x))$ makes sense for each x in X. The function $h : X \longrightarrow Z$ defined by $h(x) = g(f(x))$ is called the **composite** of the functions f and g; it is usually denoted by $g \circ f$.

Example 16. Let $X = Y = \{1, 2, 3, 4\}$ and $Z = \{1, 2, 3\}$, and let

$$f = \{(1, 1), (2, 3), (3, 1), (4, 2)\} \subseteq X \times Y,$$

and

$$g = \{(1, 2), (2, 3), (3, 1), (4, 2)\} \subseteq Y \times Z.$$

Then

$$g \circ f = \{(1, 2), (2, 1), (3, 2), (4, 3)\} \subseteq X \times Z;$$

see Figure 3 for an illustration.

Example 17. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $f(x) = x^2$, and let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $g(x) = x + 1$. Then

$$(f \circ g)(x) = f(g(x)) = f(x + 1) = (x + 1)^2 = x^2 + 2x + 1,$$

and

$$(g \circ f)(x) = g(f(x)) = g(x^2) = x^2 + 1.$$

In general $f \circ g \neq g \circ f$.

3.4 Correspondences

A **correspondence** Γ from a nonempty set X into a nonempty set Y, denoted by $\Gamma : X \rightrightarrows Y$, is a function that maps X to $\mathscr{P}(Y) \backslash \{\varnothing\}$: it assigns every element of X a *nonempty subset* of Y. In other words, for each $x \in X$, $\Gamma(x)$ is a nonempty subset of Y.[17] If Γ is single-valued, that is, $\Gamma(x)$ is a singleton for all $x \in X$, then it can be thought of as a function mapping X into Y. In economics and business, the terms "single-valued correspondence" and "function" are usually used interchangeably.

Example 18. In Figure 4, you can see the graph of function $f(x) = x^2$ and correspondence $\Gamma(x) = [0, x^2]$ at x_0. For each $x \in \mathbb{R}$, f assigns $x^2 \in \mathbb{R}_+$ to it, and Γ associates it with $[0, x^2] \subseteq \mathbb{R}_+$.

[17] The term "correspondence" is mainly used by economists; mathematicians usually refer to such an object as set-valued functions, multivalued function, or multifunction.

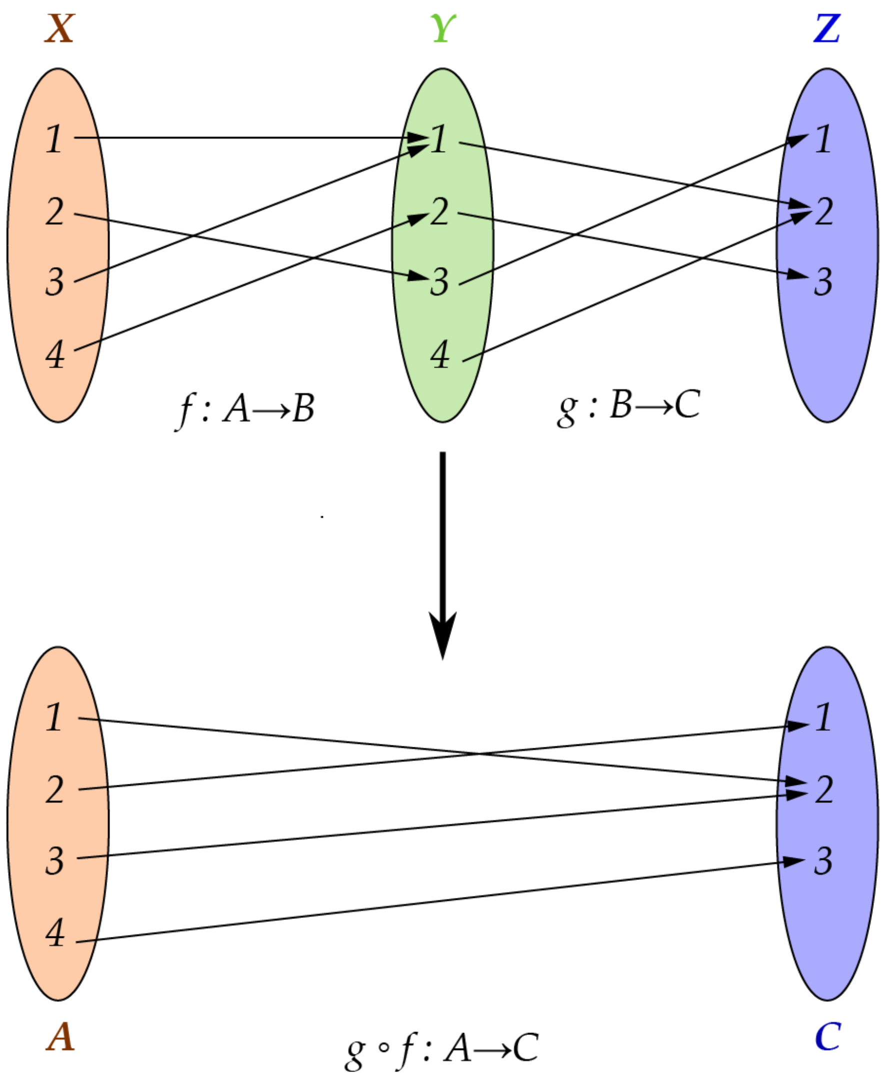


Figure 3: An example for the composition of two functions.

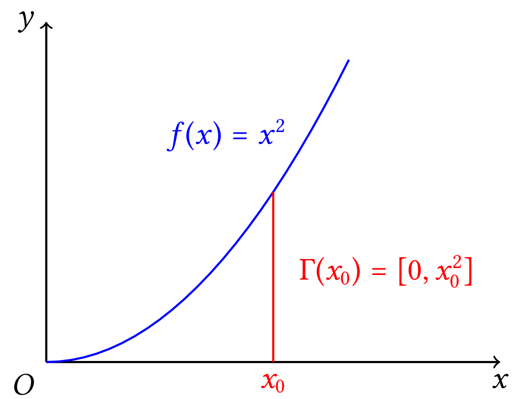


Figure 4: The graph of function $f(x) = x^2$ and correspondence $\Gamma(x) = [0, x^2]$ at x_0.

Example 19. Suppose a consumer with wealth $\iota > 0$ is interested in buying two goods, denoted x_1 and x_2. The prices of the two goods are p_1 and p_2, respectively; both p_1 and p_2 are strictly positive. Define

$$B(p_1, p_2, \iota) = \{(x_1, x_2) \in \mathbb{R}^2_+ \,:\, p_1 x_1 + p_2 x_2 \leq \iota\},$$

which is usually called the *budget set* of a consumer, is the set of all consumption bundles that are affordable to the consumer. If we treat p_1, p_2 and ι as variables, then it would be necessary to view B as a correspondence. We have $B : \mathbb{R}^3_{++} \rightrightarrows \mathbb{R}^2_+$, where

$$\mathbb{R}^3_{++} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \,:\, x_1 > 0, x_2 > 0, x_3 > 0\}.$$

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