ECN312: Intermediate Microeconomic Theory

Math Review

Kun Zhang*

June 29, 2022

These notes cover some mathematical prerequisites of this class. In the first three sections, I provide a quick review of what you should have learned in your calculus class. In Section 4 I briefly introduce some simple optimization techniques which are used intensively in the remainder of this class. I sacrifice some rigorousness for the sake of accessibility as well as expositional convenience.

1 Functions

When we talk about a function, we mean a rule that "transforms" the objects in a given set to those of another. A **function** from X to Y is a rule that assigns *each* object in X *one and only one* object in Y. In this case we write $f : X \longrightarrow Y$; call X the **domain** of f and Y is the **codomain**. Check Figure 1 to make sure that you understand the definition of a function.

1.1 Invertible Functions and Inverse

Given a function $y = f(x)$, can we always express x as a function of y, say $h(y)$? Not necessarily. If this is possible for a function f, we say that f is **invertible**; and h is the **inverse** of f. For a function f to be invertible, it has to be that for distinct objects in the domain, f assigns different objects in the codomain. To be more formal, $x \neq x'$ implies that $f(x) \neq f(x')$. As a first example, the functions shown in the upper left and the upper right panels of Figure 1 are not invertible.

More examples are provided in Figure 2. $g(x) = x^2/2$ is not invertible because $g(-2) = g(2) = 2$: if the inverse of g exists, "input" 2 corresponds to two "outputs", namely 2 and -2, which is not

*Department of Economics, Arizona State University. Email: kunzhang@asu.edu
Some material in these notes are taken from the videos by Fernando Leiva Bertran, whom I thank. I am also grateful to Marco Escobar for helpful suggestions.

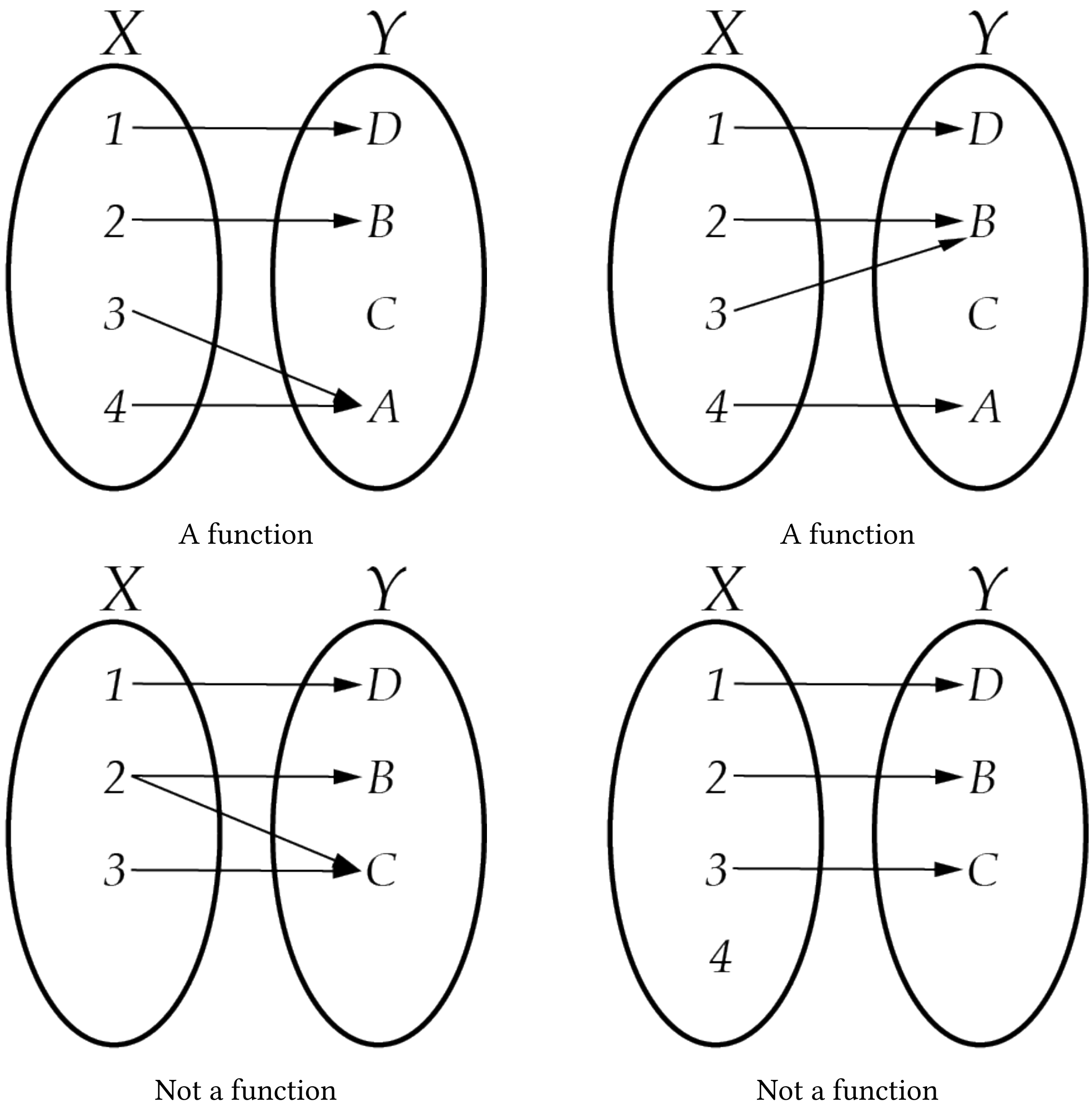
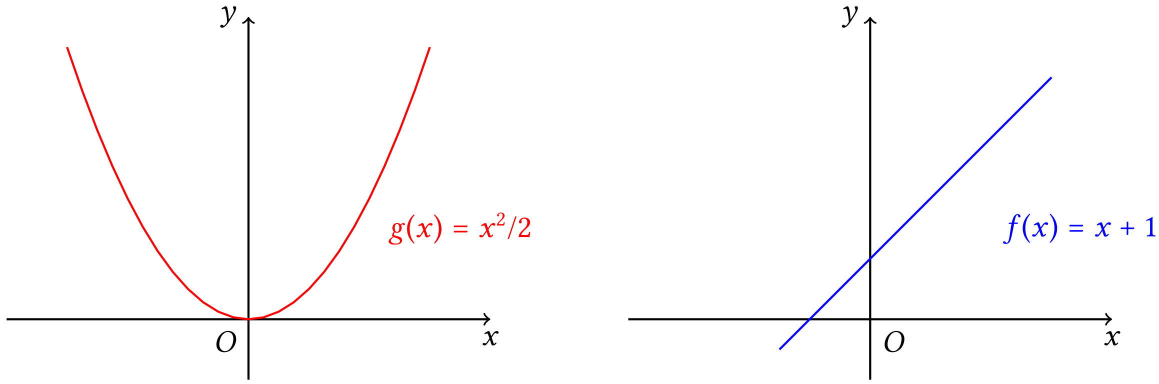


Figure 1: An illustration of the property of a function



$g(x) = x^2/2$ is not invertible $\quad\quad$ $f(x) = x + 1$ is invertible

Figure 2: Further examples on functions that are invertible or not invertible.

compatible with the definition of a function. $f(x) = x + 1$ is invertible: if $y = x + 1$, then $x = y - 1$; and hence $h(y) = y - 1$ is the inverse of f.

Now consider $j(x) = x^2/2$, but with $x > 0$. Interestingly, $j(x)$ is invertible: for $x, x' > 0$, if $x \neq x'$, then $x^2/2 \neq (x')^2/2$. Letting $y = x^2/2$ for $x > 0$, we can find $x = \sqrt{2y}$. Thus, the inverse of $j(x)$ is $k(y) = \sqrt{2y}$.

1.2 Exponentiation and Logarithm

The functions in which the variable x appears as an exponent are called **exponential functions**. We are only interested in a particular exponential function, $f(x) = e^x$, where e is a constant approximately equal to 2.71828, known as Euler's number. We abuse notation and call $f(x) = e^x$ *the* exponential function. It satisfies the following identities:

(1) $e^{x+y} = e^x e^y$,

(2) $(e^x)^y = e^{xy}$,

(3) $e^{-x} = 1/e^x$,

(4) $e^0 = 1$.

The inverse of the exponential function is called a **logarithm function**, denoted by $l(x) = \log x$.[1] The logarithm function has the following properties:

(1) $\log(xy) = \log x + \log y$,

(2) $\log(x/y) = \log x - \log y$,

[1]For those who know more about logarithm functions, some author write this function as $\log_e x$ or $\ln x$.

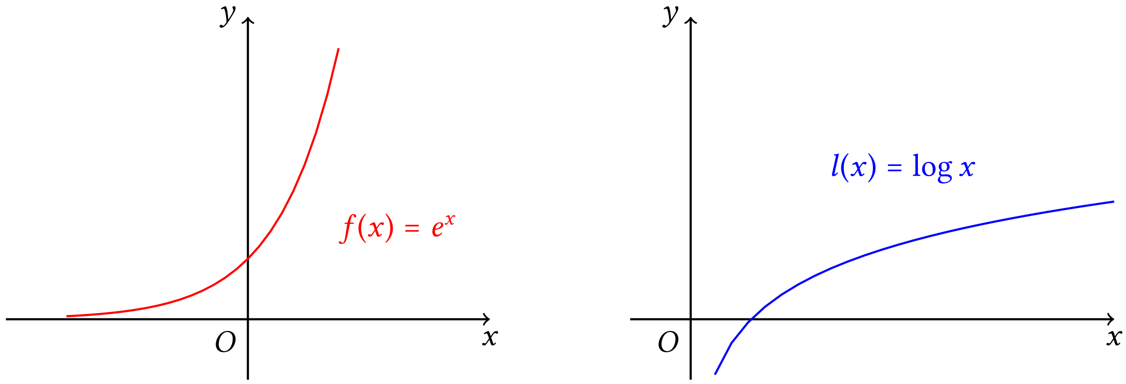


Figure 3: Graphs of the exponential function (left panel) and the logarithm function (right panel).

(3) $\log(x^y) = y \log x$,

(4) $\log 1 = 0$.

The exponential function and logarithm function are plotted in Figure 3. Note that since $f(x) = e^x > 0$ for all x, $\log x$, which is the inverse of $f(x)$, is only defined for $x > 0$.

2 Univariate Functions

2.1 From Slopes to Derivatives

We know from high school or pre-calculus that

$$\text{slope} = \frac{\text{rise}}{\text{run}}.$$

More formally, for a univariate function f, and two points x_0 and x_1, the slope between x_0 and x_1 is given by

$$\frac{f(x_1) - f(x_0)}{x_1 - x_0}.$$

The slope between x_0 and x_1 measures the "rate of change" between these two points.

Say that f is **differentiable** at x_0 if

$$\lim_{h \to 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}$$

exists. The **derivative** of f at x_0, denoted $f'(x_0)$, or $\frac{\mathrm{d}f}{\mathrm{d}x}(x_0)$, is this limit. We see that the derivative of f at x_0 is the limit of the slope between x_0 and x_1, or the rate of change, as $x_1 = x_0 + \Delta x$

approaches x_0;[2] this is illustrated in Figure 4. And geometrically, the derivative of f at x_0 is the slope of the line that tangent to the graph of f at x_0.

The function f is said to be **differentiable** if it is differentiable at all points. In this case the **derivative** of f is defined as the function $f'(x)$ that assigns each x the derivative of f at x.

In this class, if you do not know how to determine whether a function is differentiable formally, it is not that big a deal. Nonetheless, you should at least know that if a function has a jump or a kink, then it is *not* differentiable at the jump or the kink; see Figure 5 for two examples.

2.2 Rules of Taking Derivatives

For some complicated functions, computing derivatives using definition could be a tricky and daunting task. The following few results provide some "shortcuts" for it. Using the definition, we can calculate the following formulas for the derivative of specific functions, where a, n, and k are constants.

$f(x)$	$f'(x)$
k	0
kx^n	knx^{n-1}
$\log x$	$1/x$
e^x	e^x
a^x	$a^x \log a$
$\cos x$	$-\sin x$
$\sin x$	$\cos x$

We do not attempt to prove all of the derivatives in the table above; instead, we find the derivative of $f = x^3$ as an example to fix ideas. By definition, for any x, we have

$$\lim_{\Delta x \to 0} \frac{(x + \Delta x)^3 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^3 + 3x^2(\Delta x) + 3x(\Delta x)^2 - x^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{3y^2(\Delta x) + 3y(\Delta x)^2}{\Delta x}$$
$$= \lim_{\Delta x \to 0} (3x^2 + 3x(\Delta x)) = 3x^2,$$

where the first equality holds because

$$(x + \Delta x)^3 = (x + \Delta x)(x + \Delta x)^2 = (x + \Delta x)(x^2 + 2x(\Delta x) + (\Delta x)^2) = x^3 + 3x^2(\Delta x) + 3y(\Delta x)^2 - x^3.$$

Because x is arbitrary, we have $f'(x) = 3x^2$ for all x.

Next we introduce some more general rules of finding derivatives.

Result 1. *Let f and g be differentiable univariate functions. Then*

[2] Loosely, we can say that it is the rate of change *at* x_0.

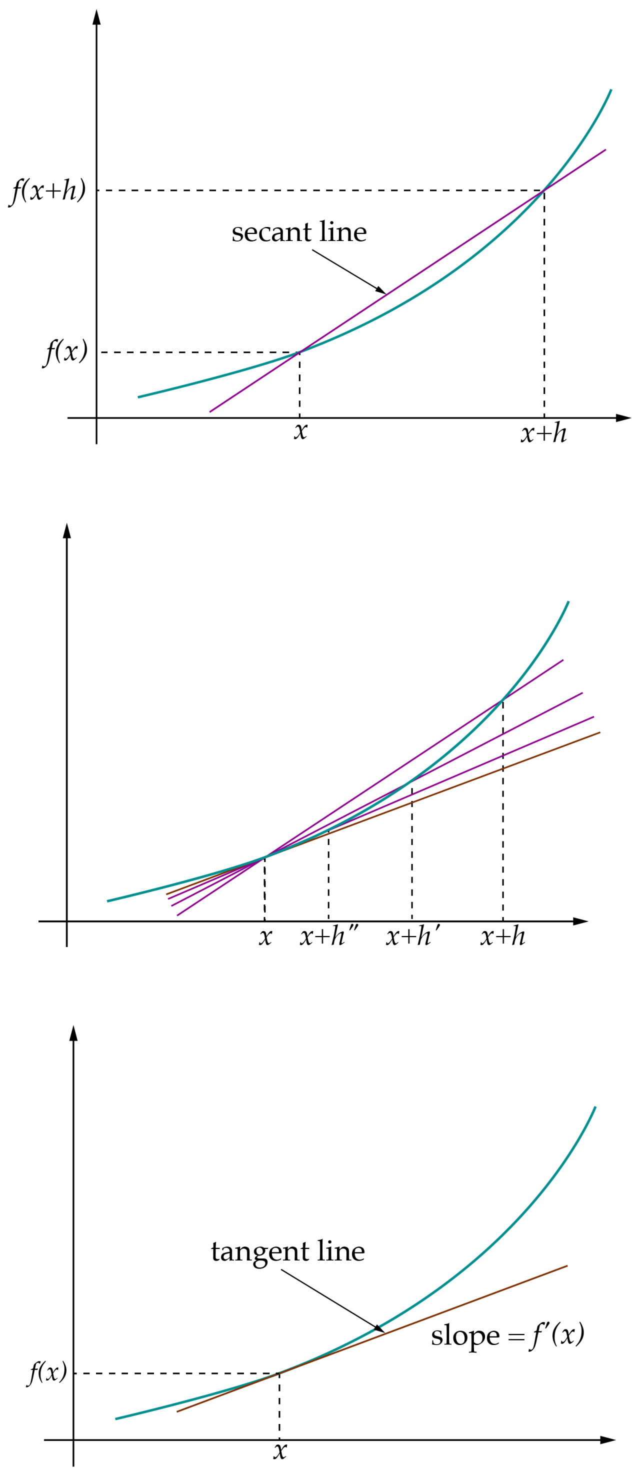


Figure 4: The derivative of a function at a point x is the limit (as Δx approaches 0) of secants to curve $y = f(x)$ determined by points $(x, f(x))$ and $(x + \Delta x, f(x + \Delta x))$; equivalently, it is the slope of the tangent line at $(x, f(x))$.

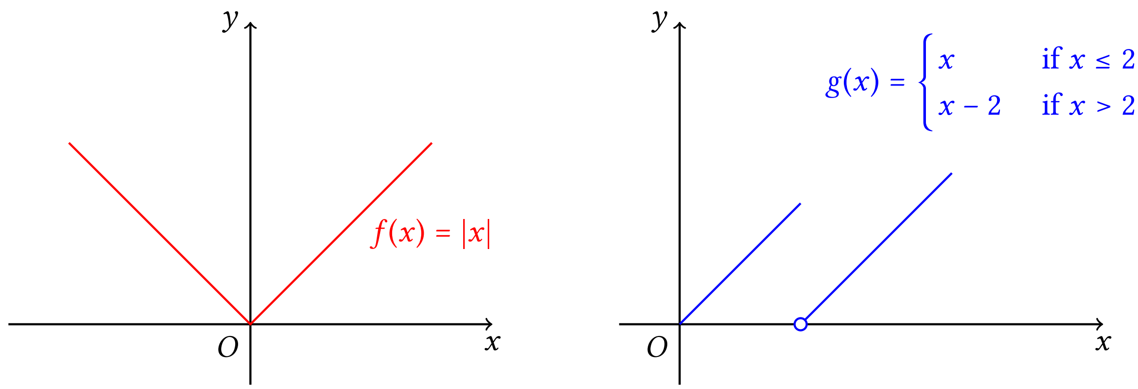


Figure 5: Examples of non-differentiable functions. $f(x)$ displayed in the left panel is not differentiable at $x = 0$ because it has a kink there; $g(x)$ plotted in the right panel is not differentiable at $x = 2$ because there is a jump.

(1) $[f(x) + g(x)]' = f'(x) + g'(x)$;

(2) $[f(x)g(x)]' = f'(x)g(x) + f(x)g'(x)$; *and*

(3) *if* $g(x) \neq 0$,

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{g(x)f'(x) - g'(x)f(x)}{g^2(x)}.$$

Example 1. Consider $f(x) = x^2(\log x + 1)$. Its derivative is given by

$$f'(x) = 2x(\log x + 1) + x^2(1/x) = 2x \log x + 3x.$$

Consider $g(x) = 1/x$. It derivative is given by

$$g'(x) = \frac{0 \cdot x - 1 \cdot 1}{x^2} = -\frac{1}{x^2}.$$

Result 2 (Chain rule)**.** *Let f and u be univariate differentiable functions. If $f(x)$ can be written as $f(u(x))$, then*

$$f'(x) = f'(u)u'(x).$$

Example 2. Consider $f(x) = e^{2x^2}$. To use the chain rule, let $u = 2x^2$. Then

$$f'(x) = f'(u)u'(x) = e^u(4x) = 4xe^{2x^2}.$$

Now consider $g(x) = \log(x^2 + 5)$. Again, let $v = x^2 + 5$. Then

$$g'(x) = g'(v)v'(x) = \left(\frac{1}{v}\right) 2x = \frac{2x}{x^2 + 5}.$$

3 Multivariate functions

Let f be a multivariate function; for expositional ease, assume that it is a function of three variables, x, y, and z. The **partial derivative** of f with respect to its x, at (x^0, y^0, z^0), is

$$\lim_{\Delta x \to 0} \frac{f(x^0 + \Delta x, y^0, z^0) - f(x^0, y^0, z^0)}{\Delta x},$$

provided that the above limit exists. The partial derivative of f with respect to y and z are analogously defined.

When the partial derivative of f with respect to x exists, it is the derivative of f with respect to x *holding all other variables fixed*; it is denoted $\frac{\partial f}{\partial x}$. Consequently, to calculate partial derivatives, we just need to take all other variables as constants, and apply the rules we introduced in Section 2.2 to x_j.

Example 3. Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be defined as

$$f(x, y) = x^2 + xy + y + 1,$$

then

$$\frac{\partial f}{\partial x} = 2x + y, \quad \text{and} \quad \frac{\partial f}{\partial y} = x + 1.$$

Let $g : \mathbb{R}^2_{++} \longrightarrow \mathbb{R}$ be defined as

$$g(x, y) = e^{xy} + y \log x + 3,$$

then

$$\frac{\partial g}{\partial x} = ye^{xy} + \frac{y}{x}, \quad \text{and} \quad \frac{\partial g}{\partial y} = xe^{xy} + \log x.$$

Result 3 (Chain rule for partial derivatives)**.** *Let f and u be multivariate differentiable functions; for expositional ease, assume that they are functions of three variables. If $f(x, y, z)$ can be written as $f(u(x, y, z))$, then*

$$\frac{\partial f}{\partial x} = f'(u)\frac{\partial u}{\partial x}, \quad \frac{\partial f}{\partial y} = f'(u)\frac{\partial u}{\partial y}, \quad \textit{and} \quad \frac{\partial f}{\partial z} = f'(u)\frac{\partial u}{\partial z}.$$

Example 4 (harder)**.** Consider

$$f(x, y) = \sqrt{3xy + \log(x + 1)}.$$

Then let $u(x, y) = 3xy + \log(x + 1)$, so $f(u) = \sqrt{u}$; consequently,[3]

$$\frac{\partial f}{\partial x} = f'(u)\frac{\partial u}{\partial x} = \left(\frac{1}{2\sqrt{u}}\right)\left(3y + \frac{1}{x+1}\right) = \frac{3y + \frac{1}{x+1}}{2\sqrt{3xy + \log(x+1)}};$$

and

$$\frac{\partial f}{\partial y} = f'(u)\frac{\partial u}{\partial y} = \frac{3x}{2\sqrt{3xy + \log(x+1)}}.$$

4 Optimization

4.1 Unconstrained Optimization

Consider a function of one variable, denote it by $f(x)$. Sometimes we are interested in finding the points at which $f(x)$ attains its maximum; write the maximization problem as $\max f(x)$. If $f(x)$ attains its maximum at x^*, we say that x^* **solves** the maximization problem $\max f(x)$, or x^* **maximizes** $f(x)$. And $f(x)$ is called the **objective function** of this maximization problem.

The following celebrated result, often called "the first-order condition", says that if a differentiable function $f(x)$ attains its maximum at x^*, then the derivative of f at x^* is 0. This is illustrated in Figure 6.

Result 4 (First-order condition, one variable)**.** *Let $f(x)$ be a differentiable function. If x^* maximizes $f(x)$, then $f'(x^*) = 0$.*

Importantly, Result 4 states that *if* x^* maximizes $f(x)$, *then* the derivative of f at x^* is 0. Not the other way around! In fact, as shown in Figure 7, we can easily find points such that the first-order condition hold but they do not solve the maximization problem. *In this class, however; so long as f is differentiable, unless otherwise specified, you can solve the maximization problem by just taking the first-order condition.*

Result 4 can be generalized to functions with many variables; to ease exposition, we only state the result for two variables. It says that, if a differentiable function $f(x, y)$ attains its maximum at (x^*, y^*), then *all* partial derivatives of f at (x^*, y^*) is 0.

Result 5 (First-order condition, two variables)**.** *Let $f(x, y)$ be a differentiable function. If (x^*, y^*) maximizes $f(x, y)$, then*

$$\frac{\partial f}{\partial x}(x^*, y^*) = \frac{\partial f}{\partial y}(x^*, y^*) = 0.$$

[3]Note that $f(u) = \sqrt{u} = u^{1/2}$; and hence $f'(u) = (1/2)u^{(1/2)-1} = (1/2)u^{-1/2} = (1/2)(1/\sqrt{u}) = 1/(2\sqrt{u})$.

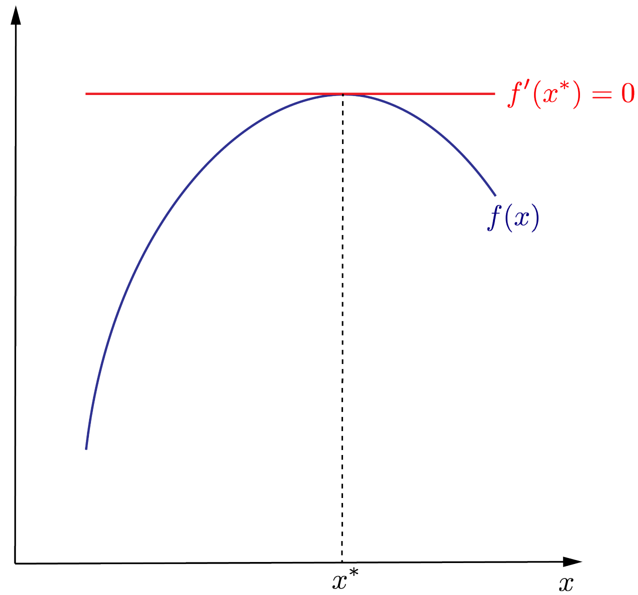


Figure 6: An illustration of the first-order condition.

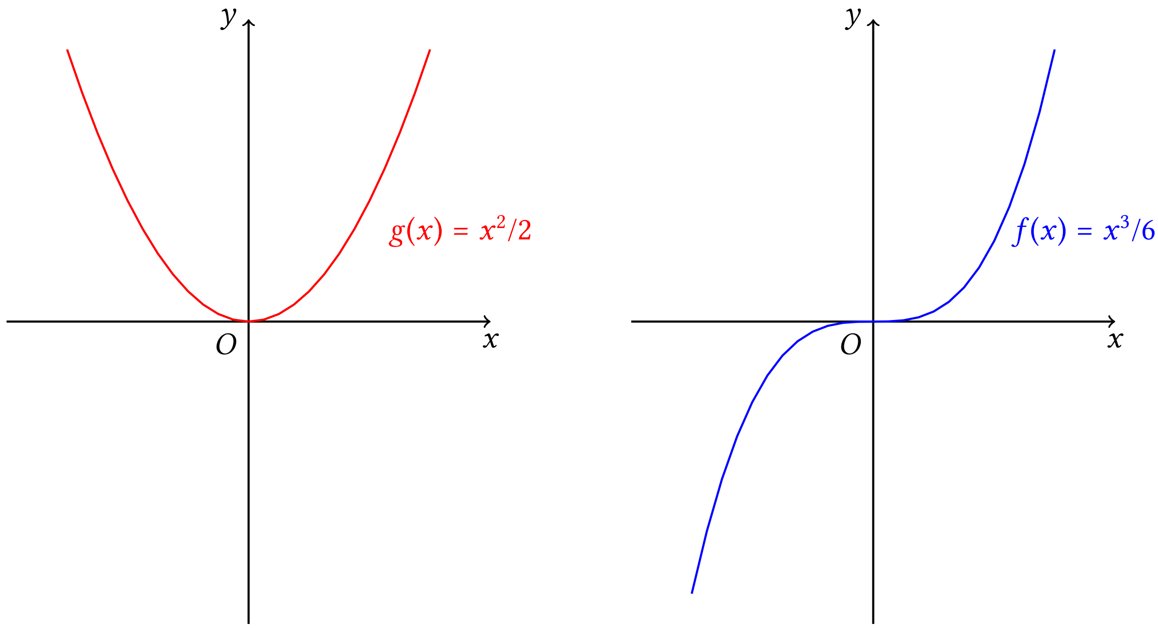


Figure 7: Points that satisfies the first-order condition may not maximize a function. The eft panel plots $g(x) = x^2/2$; $g'(0) = 0$, but clearly $x = 0$ minimizes g. In the right panel, $f(x) = x^3/6$; $f'(0) = 0$ but $x = 0$ neither maximizes nor minimizes f.

Example 5. Consider the unconstrained optimization problem

$$\max_{x,y} \; 12x + 12y - 2x^2 - y^2 + 2xy.$$

The first-order conditions are

$$\frac{\partial f}{\partial x}(x^*, y^*) = 12 - 4x^* + 2y^* = 0;$$
$$\frac{\partial f}{\partial y}(x^*, y^*) = 12 - 2y^* + 2x^* = 0.$$

Add up the two equalities above, we get $x^* = 12$. Plug this into one of the equalities above, we have $y^* = 18$. Thus, $(x^*, y^*) = (12, 18)$ solves this problem.

4.2 Constrained Optimization

Often in optimization problems, we face some constraints: Anne might want to buy the fastest car in the world, but she doesn't make that much of money; and Bob might want to go to the summit of every mountain, but for safety reasons he has to stay on the hiking trail. A constrained optimization problem can be written as

$$\begin{aligned} \max_{x,y} \quad & f(x, y) \\ \text{subject to} \quad & g(x, y) = c \end{aligned} \tag{1}$$

where c is a number.

Naturally, to deal with constrained optimization problems, we would like to transform them to unconstrained ones. One way of doing that is usually called the "substitution method": we use $g(x, y) = c$ to express y as a function of x, say $y = h(x)$, and then plug $y = h(x)$ into the objective function $f(x, y)$. To understand how it works, let us look at an example.

Example 6. Consider the constrained optimization problem

$$\begin{aligned} \max_{x,y} \quad & 12x + 12y - x^2 - y^2 + 2xy; \\ \text{subject to} \quad & x + y = 4. \end{aligned}$$

The constraint can be written as $y = 4 - x$; plugging into the objective function, we get

$$\max_{x} 12x + 12(4 - x) - x^2 - (4 - x)^2 + 2x(4 - x).$$

The first-order condition is

$$12 - 12 - 2x^* - 2(4 - x^*)(-1) + 8 - 4x^* = 0;$$

simplify, we get

$$8x^* = 16,$$

and hence $x^* = 2$. Consequently, $y^* = 4 - x^* = 2$. Therefore, the solution of the optimization problem is $(x^*, y^*) = (2, 2)$.

Note that, however; the substitution method need not work for every constrained optimization problem: in many cases, we might not be able to express y as a function of x using the constraint. See Figure 8 for an example.

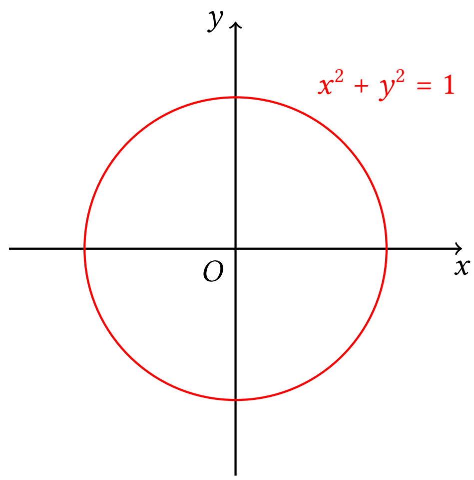


Figure 8: The equation $x^2 + y^2 = 1$ does not define y as a function of x, which makes the substitution method tricky.

The good news is, there is a more systematic way of solving a constrained optimization problem. The idea is that, by introducing an "auxiliary variable" λ, we somehow incorporate the constraint into the objective function.

For an example, consider the constrained optimization problem (1). We introduce a new function, which is called a **Lagrangian**:

$$\mathcal{L}(x, y, \lambda) = f(x, y) + \lambda[c - g(x, y)].$$

Now we solve the unconstrained optimization problem with the objective function being $\mathcal{L}(x, y, \lambda)$. The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x} = \frac{\partial f}{\partial x} - \lambda \frac{\partial g}{\partial x} = 0; \tag{2}$$

$$\frac{\partial \mathcal{L}}{\partial y} = \frac{\partial f}{\partial y} - \lambda \frac{\partial g}{\partial y} = 0; \tag{3}$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = c - g(x, y) = 0. \tag{4}$$

Note that (4) is no more than writing down the constraint once again. Because λ is an auxiliary variable and we do not really need to solve it, so we aim to eliminate it first. To this end, note that (2) and (3) can be written as

$$\frac{\partial f/\partial x}{\partial g/\partial x} = \lambda;$$
$$\frac{\partial f/\partial y}{\partial g/\partial y} = \lambda.$$

Consequently,

$$\frac{\frac{\partial f/\partial x}{\partial g/\partial x}}{\frac{\partial f/\partial y}{\partial g/\partial y}} = \frac{\lambda}{\lambda} = 1,$$

which is equivalent to

$$\frac{\partial f/\partial x}{\partial f/\partial y} = \frac{\partial g/\partial x}{\partial g/\partial y}. \tag{5}$$

Using (5) and (4), now we have two equations and two unknowns, and hence we can solve for (x^*, y^*).

For an example, consider the optimization problem in Example 6 again. Write down the Lagrangian,

$$\mathcal{L} = 12x + 12y - x^2 - y^2 + 2xy + \lambda(4 - x - y).$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial x}(x^*, y^*) = 12 - 2x^* + 2y^* - \lambda = 0;$$
$$\frac{\partial \mathcal{L}}{\partial y}(x^*, y^*) = 12 - 2y^* + 2x^* - \lambda = 0;$$
$$\frac{\partial \mathcal{L}}{\partial \lambda}(x^*, y^*) = 4 - x^* - y^* = 0.$$

The first two equations can be written as

$$12 - 2x^* + 2y^* = \lambda.$$
$$12 - 2y^* + 2x^* = \lambda;$$

and hence

$$\frac{12 - 2x^* + 2y^*}{12 - 2y^* + 2x^*} = \frac{\lambda}{\lambda} = 1 \quad \Longrightarrow \quad 12 - 2x^* + 2y^* = 12 - 2y^* + 2x^*;$$

simplify, we get $x^* = y^*$. Plug this into $4 - x^* - y^* = 0$, the solution is $(x^*, y^*) = (2, 2)$, which is the same as in Example 6.