Optimal Procurement Design: A Reduced Form Approach

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Preliminary

Abstract

Standard procurement models make the implicit assumption that the buyer knows the quality of the object at the time she procures, but in many cases, the quality is learned long after the procurement. We study procurement settings where the buyer's valuation of the good supplied depends directly on its quality, and the quality is both unverifiable and unobservable to the buyer. For a broad class of procurement problems, we identify the procurement mechanisms that maximize the buyer's expected payoff and the expected social surplus, respectively. In both cases, the optimal mechanism can be implemented by a dynamic combination of the two commonly used procurement methods: auction and negotiation. Procurement mechanisms of this kind are used in the Italian public procurement system.

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1 Introduction

Procurement plays an important role in almost all kinds of organizations. Hospitals rely on procurement to obtain medical devices (Bonaccorsi, Lyon, Pammolli, and Turchetti, 2000), and governments procure weapons and public services (Lalive, Schmutzler, and Zulehner, 2015). Firms obtain most of their production inputs as well as many components of their product lines from procurement. The most widely used procurement methods are competitive bidding and negotiating with individual suppliers.

In many procurement settings, the buyer's valuation of the good supplied depends directly on its quality, but the quality is, at least partially, unverifiable to a court, and also unobservable to the buyer at the time that the procurement contract is signed. To illustrate, suppose Anne would like to find a food service company for her workplace. While she can write in the contract that, say, all ingredients have to be organic and fair traded, and the chef has to be certified, it is certainly infeasible to require how tasty the food has to be since it cannot be verified. Before signing the contract, Anne might be able to taste a couple of dishes, but it is only possible for her to completely understand the "true quality" of the food service provider after dining there for a couple of weeks, if not a couple of months. When the quality of the good supplied is both unverifiable and unobservable, what procurement mechanisms can be optimal? How do the optimal mechanisms relate to the most common procurement methods, to wit, competitive bidding and negotiation?

To answer these questions, we revisit the optimal procurement design problem studied in Manelli and Vincent (1995), where the buyer's valuation is a function of quality, and each supplier's quality, which is one dimensional, is her private information. Following Manelli and Vincent (1995), we model competitive bidding as a second-price auction, and model negotiation as the buyer making take-it-or-leave-it offers. We also consider two objectives: maximizing the buyer's expected payoff, and maximizing expected social surplus; the latter can be relevant in government procurement design problems.[1] We focus on the case that the sellers are symmetric, that is, their qualities are independently and identically distributed.

Manelli and Vincent (1995) concern the optimality of the two commonly used procurement methods. They find that when the buyer's marginal valuation of quality is uniformly small, conducting an auction is optimal; and when the buyer's marginal valuation of quality is uniformly large, making individual offers is optimal.[2]

More complicated curvatures; however, arise naturally in many relevant procurement

[1] We argue in Section 5.2 that the main results continue to hold if the objective is a weighted average of these two.

[2] More precisely, by "uniformly large" we mean that the buyer's marginal valuation of quality uniformly dominates a certain function, and it is "uniformly small" if it is uniformly dominated by the same function.

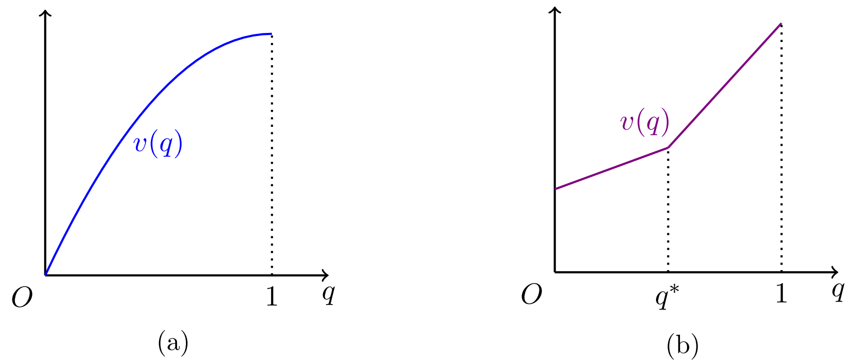

Figure 1: In panel (a), the buyer's valuation curve is strictly concave in quality; in particular, the marginal valuation is large when quality is low, and small when quality is high. In panel (b), the buyer's marginal valuation is small on $[0, q^*]$ and large on $[q^*, 1]$.

problems. For example, in many procurement settings, it is natural to assume that the buyer's valuation function looks like Figure 1a: the marginal valuation is large when quality is low, and small when quality is high; this is illustrated in Figure 1a. As another example, suppose a firm procures an input for its product. While the firm does not observe the quality of the input, they know that all else equal, the final product can be sold in the high-end market if and only if the input quality is above a certain threshold, otherwise it has to be sold in the low-end market; and the consumers in the high-end market are much more sensitive to the quality of the product.[3] Consequently, the firm's marginal valuation of quality, as shown in Figure 1b, is much larger when the quality of the input is above the quality threshold q^* than below q^*. We, instead, explicitly solve for the buyer's optimal and socially optimal procurement mechanisms.

To find the optimal procurement mechanisms, we adopt a "reduced form" approach. Specifically, we transform both buyer's optimal and socially optimal design problems into choosing an interim allocation rule that maximizes a linear functional identified by some virtual surplus.[4] Virtual surplus is a function of quality that specifies the payoff from trading with a potential seller. The interim allocation rule has to satisfy a monotonicity constraint that ensures incentive compatibility (or truthtelling), and Border's (1991) condition that guarantees feasibility. Then we apply techniques on linear optimization under a majorization constraint developed by Kleiner, Moldovanu, and Strack (2020) to characterize both

[3]For a concrete example, consider an automobile manufacturer that procures material (say steel or aluminum) for car bodies. Cars with low-quality body materials can be sold to some low-end markets where consumers mainly care about prices. To break into European or North American markets; however, a car model has to pass some crash tests to gain trust, and consumers in these markets care much more about safety.

[4]An interim allocation rule, specifies, for each reported quality of a seller, the expected probability that the buyer procures from her. The expectation is taken over all other sellers' qualities. It is also called a reduced form in the auction literature.

the buyer's optimal and socially optimal interim allocation. Unless the virtual surplus is decreasing in quality,[5] the ironing technique (Myerson, 1981; Toikka, 2011) is needed; we call the maximal intervals on which the ironed virtual surplus is flat "pooling intervals". The optimal interim allocation is constant on each of the pooling intervals, and it is otherwise strictly decreasing.

To show that a trading mechanism is optimal, we verify that it is consistent with the optimal interim allocation. On an interval where virtual surplus is decreasing, competitive bidding selects the seller with the lowest quality, and hence optimal.[6] On an interval where virtual surplus is increasing, the buyer's marginal valuation of quality must be high enough; so on this interval, she has strong quality concerns. In this case; however, competitive bidding selects the "worst" seller, and negotiation is optimal because it minimizes the competition among potential sellers. These commonly used procurement mechanisms are, in fact, "building blocks" for the optimal trading mechanism we identify: we find that it is optimal to make sellers whose qualities fall in pooling intervals prefer accepting an offer, and let all other sellers bid in an auction with a reserve price; this is made possible by carefully choosing offers and reserve prices.

Consequently, under certain regularity conditions, a dynamic combination of second-price auctions with reserve prices and sequential take-it-or-leave-it offers is optimal. Intuitively, whenever quality concerns are strong, an auction may perform poorly, so it is optimal to "locally" replace it, namely on pooling intervals, by take-it-or-leave-it offers. Outside of the pooling intervals; however, quality concerns are dominated by cost-saving motives, which makes auctions optimal. The "optimal switching points" are exactly the boundary points between pooling intervals and intervals on which the optimal interim allocation is strictly decreasing. This result allows us to identify an optimal trading mechanism in a broad class of procurement problems.

Interestingly, a similar trading mechanism can be found in the Italian public procurement system. As noted by Decarolis and Giorgiantonio (2015) and Che, Condorelli, and Kim (2018), public procurement laws in Italy allow a procurer to negotiate with potential sellers if an initial auction fails to attract a bid below the reserve price.

[5] We use "increasing" and "decreasing" in the weak sense: "strict" will be added whenever needed.

[6] Recall that when one seller wants to sell an indivisible object to potential buyers, a standard auction selects the buyer who makes the highest bid, and this buyer also has the highest valuation. In a procurement auction, potential sellers compete for the lowest bid, and thus it selects the seller with the lowest quality.

Related literature

To the best of our knowledge, procurement problems with unverifiable and unobservable qualities are first studied by Manelli and Vincent (1995).[7] They point out that in many cases, a standard procurement auction based only on price may have poor performance; instead, it can be optimal to sequentially render take-it-or-leave-it offers to the potential sellers. Manelli and Vincent (2004) show that some "hybrid mechanisms" can also be optimal in certain procurement settings. These mechanisms include making sequential offers first and conducting an auction if all these offers are rejected, two rounds of sequential offers, as well as an auction with a reserve price followed by take-it-or-leave-it offers, which are made only if no sellers meet the reserve price. Compared to these papers, our work identifies both the buyer's optimal and socially optimal procurement mechanisms in a broader class of environments.

The most closely related paper is Lopomo, Persico, and Villa (2022). These authors find the procurement mechanism that maximizes any weighted average of the expected buyer surplus and the expected social surplus, under the stronger assumption that the buyer's virtual surplus is single-peaked. The optimal mechanism, called Lowball Lottery Auction (LoLA), only differs from a standard second-price procurement auction with a reserve price in that sellers are not allowed to bid *below* a certain "floor price".[8] LoLA is arguably simpler than the optimal mechanism identified in this paper when the buyer's virtual surplus is single-peaked, but it is not clear how it generalizes beyond this environment. Furthermore, they do not impose the restriction that the buyer's expected payoff must be nonnegative.

In the mechanism design literature, this is not the only paper that features an optimal mechanism as a combination of auction and negotiation: similar mechanisms can be optimal in Che et al. (2018) and Gershkov, Moldovanu, Strack, and Zhang (2021) (see also Zhang (2018)). Although we share the same rough intuition that it is optimal to use negotiation instead whenever an auction does not work well, the driving economic forces are very different. In Che et al. (2018) negotiation is used to deter bidder collusion, and in Gershkov et al. (2021) since bidders' values are endogenously determined, negotiation can be used to increase their investment incentives. In this paper; however, negotiation is used to mitigate quality concerns whenever such concerns are strong enough, namely the marginal valuation of quality is high enough.

[7]For a thorough review of the procurement literature, we direct the readers to Che (2008).

[8]If two or more sellers bid exactly the floor price, each of them supplies the good with equal probability.

Outline

The remainder of this paper is organized as follows. Section 2 sets up and transforms the procurement design problems. Section 3 and Section 4 contain our main results on buyer's optimal and socially optimal procurement mechanisms, respectively. Section 5 discusses some assumptions and extensions. Section 6 concludes.

2 The procurement problem

The model, which is essentially the same as in Manelli and Vincent (1995, 2004), consists of one buyer and $n > 1$ symmetric potential sellers. The buyer would like to procure one unit of a product from one of the potential sellers. Seller s has private information $q_s \in [0, 1]$, and q_s's are independent and identically distributed according to a continuous density function f; we denote the corresponding cumulative distribution function by F. We also assume that $f(q) > 0$ for $q \in (0, 1]$, and $f(0) = 0$ only if

$$\lim_{q \to 0} \frac{F(q)}{f(q)} = 0.$$

We will refer to q_j as the quality of the product offered by Seller j; it can also be interpreted as Seller j's cost or reservation value.[9] Since we assume that the potential sellers are symmetric, we often suppress the subscript of a seller's quality.

All agents in our model are expected utility maximizers. If the buyer procures the good from a seller, and a transfer t is made, the seller's payoff is $t - q$. If a seller does not sell, her payoff is zero. The buyer's valuation for a good of quality q is a continuous function $v(q)$. If the buyer makes a transfer t and receives an object of quality q, her payoff is $v(q) - t$; if no trade occurs, the buyer's payoff is zero.

By the revelation principle, it suffices to focus on direct mechanisms. A direct mechanism is characterized by a pair of functions $p_s : [0, 1]^n \to [0, 1]$ and $t_s : [0, 1]^n \to \mathbb{R}$ for each seller s. If the sellers report $\mathbf{q} = (q_1, \ldots, q_n)$, the buyer procures from seller s with probability $p_s(\mathbf{q})$, and she makes transfer $t_s(\mathbf{q})$ to seller s. Because the buyer wishes to procure (at most) one unit of the product, for each $\mathbf{q} \in [0, 1]^n$, the feasibility constraint must hold:

$$\sum_{s=1}^{n} p_s(\mathbf{q}) \le 1; \tag{F}$$

(F) requires that the probability that the buyer buys from one of the potential sellers is less

[9]See Section 5.1 for a further discussion on the interpretation of q_j.

than or equal to 1.

If seller s reports q_s' and assumes that the rest of the sellers report truthfully, she would expect that the buyer procures from her with probability

$$P_s(q_s') := \int p_s\left(q_s', \mathbf{q}_{-s}\right) f^{n-1}\left(\mathbf{q}_{-s}\right) \mathrm{d}\mathbf{q}_{-s},$$

where $\mathbf{q}_{-s} = (q_1, \ldots, q_{s-1}, q_{s+1}, \ldots, q_n)$, and $f^{n-1}(\mathbf{q}_{-s}) = \prod_{k \neq s} f(q_k)$; and she would expect to receive a monetary transfer of

$$T_s(q_s') := \int t_s\left(q_s', \mathbf{q}_{-s}\right) f^{n-1}\left(\mathbf{q}_{-s}\right) \mathrm{d}\mathbf{q}_{-s}.$$

Then the expected payoff of seller s when having quality q_s and reporting q_s' is

$$\pi_s(q_s' \mid q_s) := T_s(q_s') - q_s P_s(q_s');$$

and we let $\pi_s(q_s) := \pi_s(q_s' \mid q_s)$. We say that a direct mechanism $\{p_s, t_s\}_{s=1}^n$ is **incentive compatible** if for each seller s, all $q_s' \in [0,1]$, and (almost) all $q_s \in [0,1]$,

$$\pi_s(q_s) \geq \pi_s(q_s' \mid q_s);$$

and it is **individually rational for the sellers** if $\pi_s(q_s) \geq 0$ for each seller s and $q_s \in [0,1]$. Finally, the buyer's expected payoff under direct mechanism $\{p_s, t_s\}_{s=1}^n$ is

$$\pi_b = \sum_{s=1}^n \int_{[0,1]^n} \left[v\left(q_s\right) p_s(\mathbf{q}) - t_s(\mathbf{q})\right] f^n(\mathbf{q})\, \mathrm{d}\mathbf{q}, \tag{1}$$

where $f^n(\mathbf{q}) = \prod_{s=1}^n f(q_s)$.

Lemma 1 characterizes the set of incentive compatible direct mechanisms, and also eliminates transfers from the buyer's expected payoff. The proof is standard and hence omitted.

Lemma 1. *Let $\{p_s(\cdot)\}_{s=1}^n$ be a collection of allocation functions, where $p_s : [0,1]^n \to [0,1]$, satisfying (F).*

(1) There exists a collection of transfers $\{t_s(\cdot)\}_{s=1}^n$ such that $\{(p_s(\cdot), t_s(\cdot)\}_{s=1}^n$ is incentive compatible if and only if for each $s = 1, \ldots, n$, $P_s(\cdot)$ is decreasing.

(2) For any incentive compatible direct mechanism $\{(p_s(\cdot), t_s(\cdot)\}_{s=1}^n$,

i. it is interim individually rational for the sellers if and only if $\pi_s(1) \geq 0$;

ii. and the buyer's expected payoff is given by

$$\pi_b = \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s - \frac{F\left(q_s\right)}{f\left(q_s\right)} \right] p_s(\mathbf{q}) f^n(\mathbf{q}) \, \mathrm{d}\mathbf{q} - \sum_{s=1}^{n} \pi_s(1). \tag{2}$$

To maximize the buyer's expected payoff, sellers' individual rationality constraint should bind at $q = 1$; that is, $\pi_s(1) = 0$ for all $s = 1, \ldots, n$. Then Lemma 1 allows us to write the buyer's maximization problem as

$$\begin{aligned} \max_{\{p_s\}_{s=1}^n} \quad & \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s - \frac{F\left(q_s\right)}{f\left(q_s\right)} \right] p_s(\mathbf{q}) f^n(\mathbf{q}) \, \mathrm{d}\mathbf{q} \\ \text{subject to} \quad & \text{(F)} \\ & P_s(\cdot) \text{ is decreasing for each } s = 1, \ldots, n. \end{aligned}$$

The the social surplus maximization problem can be written as

$$\begin{aligned} \max_{\{p_s\}_{s=1}^n} \quad & \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s \right] p_s(\mathbf{q}) f^n(\mathbf{q}) \, \mathrm{d}\mathbf{q} \\ \text{subject to} \quad & \text{(F)} \\ & P_s(\cdot) \text{ is decreasing for each } s = 1, \ldots, n \\ & \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s - \frac{F\left(q_s\right)}{f\left(q_s\right)} \right] p_s(\mathbf{q}) f^n(\mathbf{q}) \, \mathrm{d}\mathbf{q} \geq 0, \end{aligned}$$

where the objective function is the expected social surplus from trade, and it is without loss to set $\pi_s(1) = 0$ for all $s = 1, \ldots, n$. The inequality constraint requires that the buyer's expected payoff must be nonnegative.

2.1 Detour: Majorization

Let $f, g \in L^1(0,1)$ be decreasing. Say that f **majorizes** g, denoted by $g \prec f$, if the following two conditions hold:

$$\int_0^x g(s) \, \mathrm{d}s \leq \int_0^x f(s) \, \mathrm{d}s \quad \text{for all } x \in [0,1], \tag{3}$$

$$\int_0^1 g(s) \, \mathrm{d}s = \int_0^1 f(s) \, \mathrm{d}s. \tag{4}$$

Say that f **weakly majorizes** g, denoted by $g \prec_w f$, if (3) holds (but not necessarily (4)). In what follows we make use of some results on linear maximization under a majorization constraint developed by Kleiner et al. (2020). For readers' convenience, we include these results in Appendix A.

2.2 Transforming the problem: A reduced form approach

One novelty of this paper is that, instead of using a duality approach to characterize the optimal ex post allocation rule, we adopt a reduced form approach: we solve for the optimal interim allocation and identify a trading mechanism that implements it.

To this end, note that the buyer's expected payoff (2) can be written as, in terms of interim allocation probabilities,

$$\pi_b = \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s - \frac{F\left(q_s\right)}{f\left(q_s\right)} \right] P_s(q_s) f(q_s) \, \mathrm{d}q_s.$$

Say that the collection of interim allocations $\{P_s\}_{s=1}^{n}$, where $P_s : [0,1] \to [0,1]$ for each s, is **implementable** if there exists a collection of allocation probabilities $\{p_s\}_{s=1}^{n}$ satisfying (F) that induces $\{P_s\}_{s=1}^{n}$ as its interim allocations; that is, for each $s = 1, \ldots, n$ and all $q_s \in [0,1]$,

$$P_s(q_s) = \int p_s\left(q_s, \mathbf{q}_{-s}\right) f^{n-1}\left(\mathbf{q}_{-s}\right) \mathrm{d}\mathbf{q}_{-s}.$$

Since the sellers are symmetric, it is without loss to restrict attention to symmetric interim allocations; so we can drop the subscript s from P_s and q_s, and write P and q instead. Consequently, the buyer's expected payoff further reduces to

$$\pi_b = n \int_0^1 \left[v(q) - q - \frac{F(q)}{f(q)} \right] P(q) f(q) \, \mathrm{d}q.$$

Consider the quantile $s = F(\theta)$, we define

$$\widetilde{P}(s) := P(F^{-1}(s))$$

as the **quantile interim allocation**. Let $\widetilde{P}^*(s) = (1-s)^{n-1}$; it is not difficult to see that $\widetilde{P}^*(\cdot)$ is the quantile interim allocation of the allocation rule that always procuring from the seller with the lowest quality.

Border's (1991) celebrated theorem characterizes the set of implementable interim allocations.[10] Lemma 2 translates Border's condition into majorization terminology.[11]

[10] Maskin and Riley (1984) and Matthews (1984) also make important contribution to this result; and it is extended to asymmetric auctions by Border (2007) and Mierendorff (2011).

[11] To the best of our knowledge, Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) is the first paper

Lemma 2 (Border's condition)**.** *A decreasing symmetric interim allocation rule P is implementable if and only if the associated quantile interim allocation $\widetilde{P}(s) = P(F^{-1}(s))$ is weakly majorized by $\widetilde{P}^*$.*

To simplify notation, let $h(q) := v(q) - q$ denote the social surplus function, and let

$$g(q) := v(q) - q - \frac{F(q)}{f(q)},$$

denote the buyer's virtual surplus. By Lemma 2, the quantile interim allocation of a direct mechanism that maximizes the buyer's expected payoff can be found by solving

$$\max_{\widetilde{P} \in \Omega_w(\widetilde{P}^*)} \quad \int_0^1 g(F^{-1}(s))\widetilde{P}(s)\,\mathrm{d}s \tag{5}$$

where

$$\Omega_w(\widetilde{P}^*) = \left\{ \widetilde{P} \in L^1(0,1) : \widetilde{P} \text{ is decreasing and } \widetilde{P} \prec_w \widetilde{P}^* \right\}.$$

It is straightforward to show that a solution to problem (5) exists: the objective functional is continuous and $\Omega_w(\widetilde{P}^*)$ is compact by the Helly's selection theorem.[12]

The socially optimal interim allocation should solve the following problem:

$$\begin{aligned} \max_{\widetilde{P} \in \Omega_w(\widetilde{P}^*)} \quad & \int_0^1 h(F^{-1}(s))\widetilde{P}(s)\,\mathrm{d}s \\ \text{s.t.} \quad & \int_0^1 g(F^{-1}(s))\widetilde{P}(s)\,\mathrm{d}s \geq 0. \end{aligned} \tag{6}$$

A solution to problem (6) exists: since $\Omega_w(\widetilde{P}^*)$ is compact, the constraint set

$$\Omega_w(\widetilde{P}^*) \cap \left\{ \widetilde{P} \in L^1(0,1) : \int_0^1 g(F^{-1}(s))\widetilde{P}(s)\,\mathrm{d}s \geq 0 \right\}$$

is the intersection of a closed set and a compact set and hence also compact.

that connects Border's condition to majorization (see their Footnote 4). We omit the proof of Lemma 2 since it can be proved by slightly modifying the proof of, for example, Theorem 1 in Hart and Reny (2015) or Theorem 3 in Kleiner et al. (2020).

[12]For a complete proof of this fact, see the proof of Proposition 1 in Kleiner et al. (2020). By "compact" we mean compact in the L^1 norm topology.

3 Buyer's optimal procurement mechanisms

3.1 Buyer's optimal interim allocation

In this subsection, we solve problem (5), and in Section 3.2 we identify a trading mechanism that implements the solution we find. To make sure that the monotonicity constraint holds, the ironing technique (Myerson, 1981; Toikka, 2011) may be required. Define $\widetilde{g}(s) = g(F^{-1}(s))$, and let

$$G(s) := \int_0^s \widetilde{g}(x)\mathrm{d}x,$$

and let $\overline{G}$ be the concave hull of G:

$$\overline{G}(x) := \sup\{y : (x, y) \in \mathrm{co}(G)\},$$

where $\mathrm{co}(G)$ is the convex hull of the graph of G. Equivalently, $\overline{G}$ is the pointwise smallest upper semicontinuous and concave function that lies above G (see, for example, Kamenica and Gentzkow, 2011). Let $\overline{g} = \overline{G}'$.[13] Adopting the convention that $\sup \varnothing = 0$, we define

$$\overline{S} := \sup\{s \in [0, 1] : \overline{g}(s) \geq 0\}. \tag{7}$$

Now we are ready to state the result that identifies the optimal interim allocation rule.

Theorem 1. *If there exists a collection of disjoint intervals $[\underline{s}_i, \bar{s}_i)$ indexed by $i \in \mathcal{I}$, where $[\underline{s}_i, \bar{s}_i) \subseteq [0, \overline{S}]$ for each $i \in \mathcal{I}$, such that*

- *$\overline{G}$ is affine on $[\underline{s}_i, \bar{s}_i)$ for each $i \in \mathcal{I}$, and*
- *$\overline{G} = G$ on $[0, \overline{S}] / \bigcup_{i \in \mathcal{I}} [\underline{s}_i, \bar{s}_i)$,*

then the interim allocation

$$\hat{P}(s) = \begin{cases} (1-s)^{n-1} & \text{if } s \in [0, \overline{S}] / \bigcup_{i \in \mathcal{I}} [\underline{s}_i, \bar{s}_i)\,, \\ \frac{\int_{\underline{s}_i}^{\bar{s}_i} (1-s)^{n-1}\,\mathrm{d}s}{\bar{s}_i - \underline{s}_i} & \text{if } s \in [\underline{s}_i, \bar{s}_i)\,, \\ 0 & \text{if } s \in (\overline{S}, 1] \end{cases} \tag{8}$$

is optimal.

Theorem 1 follows from the results concerning maximizing linear functionals under a majorization constraint in Appendix A; the proof is relegated to Appendix B.1. To grasp

[13] Because $\overline{G}$ is concave, it is differentiable almost everywhere. At points where it is not differentiable, we define $\overline{g}$ as the right derivative.

some intuition, consider an auxiliary problem

$$\max_{\widetilde{P}\in\Omega_w(\widetilde{P}^*)} \int_0^1 \overline{g}(s)\widetilde{P}(s)\,\mathrm{d}s. \tag{9}$$

Observe that for $s > \overline{S}$, the ironed virtual surplus is negative, and thus it is not desirable for the buyer to trade. Consequently, any solution $\breve{P}$ of problem (9) must satisfy $\breve{P}(s) = 0$. For $s \leq \overline{S}$; however, to maximize the objective, we would like to set $\widetilde{P}$ higher at points that $\overline{g}$ takes higher value. Then since $\overline{g}$ is decreasing, we want to set $\widetilde{P}(s)$ larger when s is smaller. We see from (3) that no $\widetilde{P} \in \Omega_w(\widetilde{P}^*)$ attains larger value than $\widetilde{P}^*$ itself when s is small; thus, $\widetilde{P}^*$ solves the auxiliary problem (9). Moreover, on each $[\underline{s}_i, \overline{s}_i)$, since $\overline{G}$ is affine, $\overline{g}$ is constant, so only the mean of the interim allocation on that interval matters. Consequently, putting

$$\hat{P}(s) = \mathbb{E}\left[\widetilde{P}^*(t) \;\middle|\; t \in [\underline{s}_i, \overline{s}_i)\right] = \frac{\int_{\underline{s}_i}^{\overline{s}_i} (1-s)^{n-1}\,\mathrm{d}s}{\overline{s}_i - \underline{s}_i}$$

on $[\underline{s}_i, \overline{s}_i)$ for each $i \in \mathcal{I}$ and let $\hat{P} = \widetilde{P}^*$ otherwise would make $\hat{P}$ a solution of (9). In fact, if $\hat{P}$ solves the auxiliary problem (9), it also solves the original problem if and only if it is constant on $[\underline{s}_i, \overline{s}_i)$ for each $i = 1, \ldots, M$.[14] Therefore, interim allocation $\hat{P}$ defined by (8) solves problem (5), and is thus optimal.

Observe that ironing is required whenever the buyer's virtual surplus function has an increasing region, that is, there exists an interval $(x, y) \subseteq [0, 1]$ such that $g(q)$ is increasing on (x, y). While it might be standard to assume that F/f is increasing (so $-F/f$ is decreasing), requiring that $g(q)$ is decreasing on $[0, 1]$ also imposes strong restrictions on $v(q)$: roughly speaking, the buyer's marginal valuation of quality cannot be too high anywhere on the unit interval.[15] Therefore, in our problem, the ironing technique could be unavoidable because of the nature of certain procurement settings.

3.2 Implementation

To find a buyer's optimal procurement mechanism, we need to find some trading mechanisms that implement the allocation rule the optimal interim allocation rule $\hat{P}(s)$ that we identified in Theorem 1. To gain some ideas on how to proceed, it is helpful to look at the two most straightforward cases: the virtual surplus g is either increasing or decreasing. In fact, the

[14] This is the pooling property of Myerson (1981) and Toikka (2011); it states that for all open intervals $I \subseteq [0, 1]$, $H(s) < \overline{H}(s)$ for all $s \in I$ implies that $\hat{P}$ must be constant on I.

[15] If we further assume that both v and f are differentiable, $g(q)$ is decreasing if and only if $v'(q) \leq 1 + (F(q)/f(q))'$ for all $q \in [0, 1]$. For example, when qualities are uniformly distributed on $[0, 1]$, $g(q)$ is decreasing if and only if $v'(q) \leq 2$ for all $q \in [0, 1]$.

optimal mechanisms in these two cases are going to be the building blocks of our (general) optimal trading mechanism.

If g is increasing, so is $\widetilde{g}$;[16] consequently, $\overline{G}$ is affine on the entire unit interval. Then whenever $\overline{S} > 0$, Theorem 1 implies that the solution to problem (5) is a constant function

$$\hat{P} = \int_0^1 \widetilde{P}^*(s)\ \mathrm{d}s = 1/n,$$

and $\hat{P} = 0$ otherwise. The optimal interim allocation, in this case, can be implemented by a take-it-or-leave-it offer of 1 to an arbitrarily selected seller: each seller is selected with probability $1/n$, and whenever a seller is selected, she sells her good with probability 1.

If instead g is decreasing, we must have $\widetilde{g} = \overline{g}$ by construction. Again by Theorem 1, an optimal interim allocation is

$$\hat{P}(s) = \begin{cases} (1-s)^{n-1} & \text{if } s \leq \overline{S}, \\ 0 & \text{otherwise,} \end{cases}$$

which can be implemented by a second-price (procurement) auction with a reserve price $F^{-1}(\overline{S})$: in equilibrium, a seller with quality quantile s bids her quality $q = F^{-1}(s)$, and she sells her good if and only if all other $n-1$ sellers' qualities are above q. If all bids are above $F^{-1}(\overline{S})$, the buyer does not procure from any of the potential sellers.

The discussion above is summarized in Proposition 1; these results essentially replicate the symmetric and buyer's optimal case of Corollary 2 and 3 of Manelli and Vincent (1995).

Proposition 1. *Suppose $\overline{S} > 0$, so trade can be mutually beneficial under incomplete information, then*

(1) if g is increasing, then a take-it-or-leave-it offer of 1 to a random seller is optimal;

(2) if g is decreasing, then a second-price auction with reserve price $F^{-1}(\overline{S})$ is optimal.

Note that the virtual surplus $g(q) = v(q) - q - F(q)/f(q)$ measures the difference between the buyer's valuation and the procurement cost. For cleaner intuition, let us assume for now that both $v(q)$ and $q + F(q)/f(q)$ are increasing on $[0, 1]$.[17] While a procurement auction exploits competition among potential sellers, the competition based solely on price selects the seller with the lowest quality. If the buyer's virtual surplus is decreasing in quality, her cost

[16] Since $f > 0$, F is strictly increasing, and so is F^{-1}; hence monotonicity of g is equivalent to monotonicity of $\widetilde{g}$.

[17] Proposition 1 holds without these assumptions. It is reasonable, though, to assume that the buyer's valuation is increasing in quality, and the other assumption is rather standard.

of procurement decreases more than valuation when selecting a seller with lower quality, in which case an auction is indeed optimal. If the buyer's virtual surplus is increasing; however, procuring from the seller selected by an auction becomes highly undesirable. This is because when quality decreases, the valuation decreases more than the cost of procurement. A take-it-or-leave-it offer to a randomly selected seller is optimal in this case since by doing this the buyer can completely avoid competition among sellers.

Example 1 (Che, 2008)**.** Suppose there are two sellers each with quality q drawn independently and uniformly from $[0, 1]$,[18] and the buyer's valuation function is $v(q) = 3q$. Then $g(q) = v(q) - q - q = q$, and $\overline{g}(q) = 1/2$ for all $q \in [0, 1]$.[19] Consequently, the optimal interim allocation is

$$\hat{P} = \int_0^1 (1 - q)\,\mathrm{d}q = 1/2;$$

so it is optimal for the buyer to randomly select a seller and tender a take-it-or-leave-it offer of 1. The resulting expected buyer's payoff is 1/2, while a standard auction only yields 1/3 for the buyer in expectation.[20] ♢

If the buyer's valuation does not directly depend on the quality, so $v(q) = \overline{v}$, for an auction to be optimal, it suffices to assume that F/f is increasing, which is a rather standard assumption in the mechanism design literature. In our problem; however, the shape of $v(q)$ also matters, and it is pinned down by the nature of the procurement problem. In particular, for an auction to be optimal, it suffices to further assume that the buyer values marginal quality less than the potential sellers.

Corollary 1. *Suppose $\overline{S} > 0$, $v(q) - q$ is decreasing, F is twice continuously differentiable, and both F and $1-F$ are log-concave,*[21] *then a second-price auction with reserve price $F^{-1}(\overline{S})$ is optimal.*

Proof. Since both F and $1-F$ are log-concave, $-F(q)/f(q)$ must be decreasing.[22] Then g can be written as the sum of two decreasing functions, hence also decreasing. By Proposition 1, a second-price auction with reserve price $F^{-1}(\overline{S})$ is always optimal. □

[18] When the qualities are uniformly distributed, we have $s = q$; in such cases, we do not need the change-of-variables—we iron g directly.

[19] To get $\overline{g}$, we first calculate $G(q) = \int_0^1 g(q)\,\mathrm{d}q = q^2/2$. Then since G is strictly convex on $[0, 1]$, its concave hull is obtained by "connecting" $(0, G(0))$ and $(1, G(1))$: we get $\overline{G}(q) = q/2$. Hence, $\overline{g}(q) = \overline{G}'(q) = 1/2$.

[20] Let us consider a second-price auction: the sellers bid their qualities, and the buyer procures from the winning seller and pays the bid of the losing seller. So the buyer's expected payoff is $\mathbb{E}[3\min\{q_1, q_2\} - \max\{q_1, q_2\}] = 3\mathbb{E}[\min\{q_1, q_2\}] - \mathbb{E}[\max\{q_1, q_2\}] = 3(1/3) - 2/3 = 1/3$.

[21] By Theorem 1 and Theorem 3 in Bagnoli and Bergstrom (2005), a sufficient condition for F and $1 - F$ being log-concave is that f is log-concave.

[22] To see this, differentiate to obtain $(-F/f)' = -1 - (-Ff'/f^2)$. When $f'(q) \leq 0$, because $1 - F$ is log-concave, $-Ff'/f^2 \geq (1 - F)f'/f^2 \geq -1$; and if $f'(q) > 0$, $-Ff'/f^2 \geq -1$ since F is log-concave. Thus, $(-F/f)' \leq 0$.

The next step we take is to understand the "simplest combinations" of the two most straightforward cases we studied above. The natural candidates are that g is either single-peaked or single-dipped. Proposition 2 and Proposition 3 concern these two cases, respectively.

Proposition 2. *Suppose g is single-peaked, that is, there exists $\breve{q} \in [0,1]$ such that g is increasing on $[0, \breve{q}]$ and decreasing on $[\breve{q}, 1]$. Then there exists $\hat{s} \in [0,1]$ such that $\overline{g}(s)$ is constant on $[0, \hat{s}]$ and decreasing on $[\hat{s}, 1]$. If $\overline{S} > 0$, the following trading mechanism is optimal:*

- *randomly label the sellers to be Seller 1, ..., n;*
- *after that, a take-it-or-leave-it offer p_1 is made to Seller 1;*
- *if it is rejected by Seller 1, a take-it-or-leave-it offer $p_2 > p_1$ is made to Seller 2;*
- *...;*
- *if it is rejected by Seller $n-1$, a take-it-or-leave-it offer $p_n > p_{n-1}$ is made to Seller n;*
- *if again rejected, a second-price auction with reserve price $F^{-1}(\overline{S})$ is conducted.*

Proof. The statement on the shape of the ironed quantile virtual surplus $\overline{g}$ follows since g is single-peaked: the quantile virtual surplus $\widetilde{g}$ is thus increasing on $[0, F(\breve{q})]$ and decreasing on $[F(\breve{q}), 1]$. Consequently, G is convex on $[0, F(\breve{q})]$ and concave on $[F(\breve{q}), 1]$; so by taking the concave hull, the "lower part" must be affine. More specifically, it suffices to set

$$\hat{s} = \sup \left\{ s \in [0,1] : \overline{G}(s) > G(s) \right\};$$

so $\overline{G}$ is affine on $[0, \hat{s}]$ and coincides with G otherwise.

Then the shape of $\overline{g}$ implies that if $\overline{S} > 0$, then $\overline{S} \geq \hat{s}$; hence by Theorem 1 the optimal interim allocation takes the form of

$$\hat{P}(s) = \begin{cases} \frac{\int_0^{\hat{s}} (1-s)^{n-1} \mathrm{d}s}{\hat{s}} & \text{if } s \leq \hat{s}, \\ (1-s)^{n-1} & \text{if } \hat{s} < s \leq \overline{S}, \\ 0 & \text{if } s > \overline{S}. \end{cases} \tag{10}$$

In Appendix B.2 we construct sequential offers $p_1, p_2, \ldots, p_n$ such that the following strategy is an equilibrium of this game: sellers with valuations $q \leq F^{-1}(\hat{s})$ accept the take-it-or-leave-it offer whenever it is tendered to her; and if an auction is conducted, all sellers submit a bid equals to her valuation.[23]

[23] Note that an auction is only conducted if no seller has $q \leq F^{-1}(\hat{s})$.

In equilibrium, the seller who is labeled as Seller 1 gets the offer for sure, and Seller 2 gets the offer with probability $1 - \hat{s}$, since this event only happens if Seller 1's type is above $F^{-1}(\hat{s})$; and Seller 3 gets the offer with probability $(1 - \hat{s})^2$, and so on. Then the average probability before the draw is

$$\frac{\sum_{j=1}^{n}(1-\hat{s})^{j-1}}{n} = \frac{1-(1-\hat{s})^n}{n\hat{s}} = \frac{\int_0^{\hat{s}}(1-s)^{n-1}\,\mathrm{d}s}{\hat{s}}, \tag{11}$$

where the first equality follows from the algebraic identity

$$a^n - b^n = (a-b)\sum_{k=1}^{n} a^{k-1}b^{n-1-k};$$

set $a = 1 - \hat{s}$ and $b = 1$ and rearrange, we get the desired expression. Therefore, the interim allocation of the trading mechanism is exactly (10). $\square$

To understand Proposition 2, it is useful to consider the hypothetical situation where each seller's quality is in $[F^{-1}(\hat{s}), 1]$ first. Following the same argument as the paragraph just before Proposition 1, we see that $\hat{P}\big|_{[\hat{s},1]}$ can be implemented by an auction with reserve price $F^{-1}(\overline{S})$.[24] In particular, this reserve price guarantees that all sellers with $q \in [F^{-1}(\hat{s}), F^{-1}(\overline{S})]$ submit a bid, and no seller with $q > F^{-1}(\overline{S})$ wants to participate in the auction.

Now consider the entire interval $[0, 1]$. Because $\hat{P}$ is constant on $[0, \hat{s}]$, given part (1) of Proposition 1, intuitively it should be optimal to make sellers with $q \leq F^{-1}(\hat{s})$ willing to accept an offer, and let other sellers bid in an auction. Importantly, if a seller turns down the offer made to her, which means that this seller has $q > F^{-1}(\hat{s})$, the buyer should approach other sellers sequentially until either a seller accepts an offer or all sellers turn the offers down. Only in the latter case, the buyer should conduct an auction since she is assured that none of the potential sellers have $q \leq F^{-1}(\hat{s})$, so the argument in the previous paragraph applies.

Because the potential sellers are symmetric, it is natural to consider randomly labeling them to be Seller $1, 2, \ldots, n$. Then we can "calibrate" the offers such that for each seller, no matter what number she gets $(1, 2, \ldots, n)$, takes the offer when she has $q \in [0, F^{-1}(\hat{s})]$, and chooses to wait for the potential opportunity of bidding in the auction otherwise. Importantly, for any Seller j and k with $j < k$, we must have $p_j < p_k$ since Seller j knows that if she turns the offer p_j down, there are more potential sellers "waiting ahead", so there is a smaller probability that she would have the chance to participate in the auction. Indeed,

[24] For a function $f : [0, 1] \to \mathbb{R}$ and $[a, b] \subseteq [0, 1]$, we denote the restriction of f to $[a, b]$ by $f\big|_{[a,b]}$.

(11) shows that the interim allocation probability of any seller with $q \in [0, F^{-1}(\hat{s})]$ is exactly given by $\hat{P}(s) = \int_0^{\hat{s}} (1-s)^{n-1}\,\mathrm{d}s/\hat{s}$.

Although the assumption that g is single-peaked can be restrictive, Proposition 2 is useful in many economically meaningful environments. Lemma 3 in Online Appendix D identifies a set of conditions on primitives that guarantees that g is single-peaked. In particular, if the buyer's marginal valuation of quality is (weakly) diminishing, and the quality distribution satisfies an easy-to-check condition, we can show that g is single-peaked.

Corollary 2. *Suppose $v(q)$ is concave, and F/f is convex in q, then $g(q)$ is single-peaked. Consequently, if $\overline{S} > 0$, the optimal interim allocation can be implemented by a round of sequential offers followed by an auction.*

Proof. Note that both $v(q) - q$ and $-F(q)/f(q)$ are concave under our assumptions, so their sum $g(q)$ is also concave and hence single-peaked. The last assertion follows directly from Proposition 2. □

The assumption on quality distribution in Corollary 2 might not be standard, but it is satisfied by many familiar distributions with bounded supports, including power distributions,[25] (truncated) exponential distributions, and Beta distributions with both parameters greater than or equal to 1.

Example 2 (Diminishing marginal valuation)**.** Suppose $v(q) = -2q^2 + 4q$, and q is uniformly distributed on $[0, 1]$; for simplicity assume that there are two potential sellers. Then $g(q) = v(q) - 2q = -2q^2 + 2q$, which is single-peaked; the ironed virtual surplus $\overline{g}$ is flat on [0,3/4] and coincides with g on $[3/4, 1]$. g and $\overline{g}$ are plotted in Figure 2a. So by Theorem 1, the optimal interim allocation is given by

$$\hat{P}(q) = \begin{cases} 5/8 & \text{if } q \leq 3/4, \\ 1 - q & \text{if } q > 3/4, \end{cases}$$

which is shown in Figure 2b. Then by Proposition 2, the following trading mechanism is optimal:

- randomly label the sellers to be Seller 1 and Seller 2;
- a take-it-or-leave-it offer $p_1 = 0.78$ is made to Seller 1;
- if it is rejected by Seller 1, a take-it-or-leave-it offer $p_2 = 0.88$ is made to Seller 2;

[25] The CDF of a power distribution takes the form of $F(x) = x^\alpha$, where $\alpha > 0$. When $\alpha = 1$, we get the uniform distribution.

- if it is rejected by Seller 2, a second-price auction without a reserve price is conducted.

Observe that $p_1 < p_2$: this is because when Seller 1 rejects the offer made to her, she anticipates that the auction happens with probability 1/4, but Seller 2 knows that by rejecting the offer the auction occurs with probability 1. Therefore, it is "more difficult" to make Seller 2 prefer accepting the offer than Seller 1. $\diamondsuit$

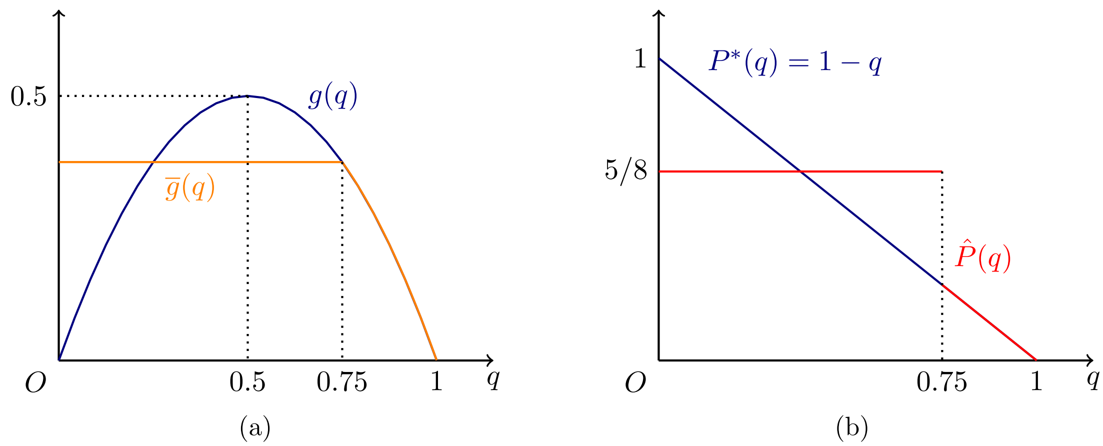

Figure 2: In panel (a), the blue curve is the buyer's virtual surplus $g(q)$, and the orange curve is the ironed virtual surplus $\overline{g}$; and in panel (b), the blue curve is $P^*(q) = 1 - q$ that appears in Border's condition, and the red curve is the optimal interim allocation $\hat{P}(q)$.

Proposition 3. *Suppose g is continuous and single-dipped, that is, there exists $\hat{q} \in [0, 1]$ such that g is decreasing on $[0, \hat{q}]$ and increasing on $[\hat{q}, 1]$. Then there exists $\tilde{s} \in [0, 1]$ such that $\overline{g}(s)$ is decreasing on $[0, \tilde{s}]$ and constant on $[\tilde{s}, 1]$. If $\overline{S} > 0$, and suppose $0 < \tilde{s} < \overline{S}$,[26] then it is optimal to run a second-price auction with a reserve price $R < 1$ first, and if no one meets the reserve price, a take-it-or-leave-it offer of 1 would be tendered to an arbitrarily selected seller.*

Since g is single-dipped, the quantile virtual surplus $\widetilde{g}$ must be decreasing on $[0, F(\hat{q})]$ and increasing on $[F(\hat{q}), 1]$. Consequently, G is concave-convex, so by taking the concave hull, an "upper part" must be affine. Let $\tilde{s}$ be defined by

$$\tilde{s} = \inf \left\{ s \in [0, 1] : \overline{G}(s) > G(s) \right\},$$

so $\overline{G}$ coincide with G on $[0, \tilde{s}]$ and it is affine otherwise. Recall that we define $\overline{S}$ as the highest quantile that $\overline{g}$ is strictly positive (see Equation (7)). The shape of $\overline{g}$ implies that if

[26] By Proposition 1, if $\tilde{s} = 0$, $\overline{g}$ is flat, so a take-it-or-leave-it offer of 1 to an arbitrarily selected seller is optimal; and if $\tilde{s} \geq \overline{S}$, a second-price auction with a reserve price $F^{-1}(\overline{S})$ is optimal.

$\overline{S} \geq \tilde{s} > 0$, then $\overline{S} = 1$; hence the optimal interim allocation is

$$\hat{P}(s) = \begin{cases} (1-s)^{n-1} & \text{if } s \leq \tilde{s}, \\ \frac{\int_{\tilde{s}}^{1} (1-s)^{n-1} \mathrm{d}s}{1-\tilde{s}} & \text{otherwise.} \end{cases} \tag{12}$$

The proof of optimality of the trading mechanism listed in Proposition 3 is relegated to Appendix B.3. It proceeds in a similar way to the proof of Proposition 2: instead of choosing the right prices offered in the sequential offer phase, the "key choice variable" in this proof is the reservation price of the auction. We show that a carefully chosen reservation price makes all sellers with quality quantile less than $\tilde{s}$ prefer to participate in the auction, and all sellers with quality quantile greater than $\tilde{s}$ to wait for the chance of an offer. Therefore, the trading mechanism proposed in Proposition 3 implements the optimal interim allocation (12).

Since we are able to find the buyer's optimal mechanism when g has one "peak" or "trough", it is natural to ask if we can use the techniques developed above to tackle a large class of problems, where g has finitely many peaks and troughs. The answer is yes.

A function h is said to be **regular** if there exists a finite partition of $[0, 1]$ into intervals such that h is either increasing or decreasing on each of the partition elements. If h is differentiable, then it is regular if and only if it changes sign a finite number of times. Regularity, as defined above, formalizes the intuitive idea of finitely many "peaks" and "troughs". Theorem 2 shows that a buyer's optimal procurement mechanism can always be implemented by alternating between sequential offers and second-price auctions (with reserve prices whenever necessary).

Theorem 2. *Suppose the virtual surplus g is regular, and $\overline{S} > 0$. Then the buyer's optimal interim allocation rule we identified in Theorem 1 can be implemented by a dynamic trading mechanism combining second-price auctions (with a reserve price whenever necessary) and sequential take-it-or-leave-it offers is optimal.*

Suppose $\overline{S} > 0$. Because g is regular, we can find a partition of $[0, 1]$ into intervals such that g is either increasing or decreasing on each of the $K < \infty$ partition elements. By Theorem 1, there are $M \leq K$ pooling intervals, on each of which the optimal interim allocation rule $\hat{P}$ is constant and nonzero; and there are $L \leq M + 1$ intervals on which $\hat{P} = \widetilde{P}^* = (1-s)^n$, we call them "non-pooling intervals". Therefore, $[0, \overline{S}]$ can be partitioned into $M + L$ disjoint intervals, where $\hat{P}$ is either constant or equals to $\widetilde{P}^*$ on each of them. In Appendix B.4 we show that, by carefully choosing the offer prices and reserve prices for the auctions, we can find a trading mechanism with L rounds of auctions and M rounds of sequential offers.

Take-it-or-leave-it offers are used if and only if the virtual surplus has an increasing region: in this region, the buyer has strong quality concerns; and by utilizing a pooling interval containing this region, she mitigates such concerns by abandoning price competition completely. Specifically, she tailors a set of offers such that only sellers in this interval would accept. On non-pooling intervals; however, auctions can be efficiently used: the buyer leverages competition from the sellers to maximize her payoff; reserve prices are set so that only sellers whose qualities fall into a certain non-pooling interval participate. Loosely speaking, it is optimal to "locally replace" auction with offers in the regions where quality concerns are strong and use auctions otherwise. It is worth noting that, in our problem, ironing is not just a technical curiosity: it helps us to determine the cutoff qualities that the buyer should optimally switch between auction and offers, namely the boundary points between pooling and non-pooling intervals.

To find the optimal offer prices and reserve prices, we start from the "last" of the $M + L$ intervals and reason backward. To simplify the argument, assume $\overline{S} = 1$; and for concreteness, let us focus on the special case shown in Figure 3. Since the last interval $[\underline{s}_M, 1]$ is a pooling interval, the buyer can randomly select a seller and render a take-it-or-leave-it offer of 1; any seller whose quality quantile falls in this interval would accept this offer. To implement an auction on the non-pooling penultimate interval $[\overline{s}_{M-1}, \underline{s}_M]$, the buyer can choose a reserve price like in Proposition 3 so that all sellers with quality quantile in $[0, \underline{s}_M]$ would prefer to bid in this auction than wait for the chance of a take-it-or-leave-it offer of 1, and the other way around for $s \in (\underline{s}_M, 1]$. For the third to last interval $[\underline{s}_{M-1}, \overline{s}_{M-1}]$, which is a pooling interval, the buyer can design a sequential offer scheme like in Proposition 2, so that regardless of the order of being offered, any seller with $s \in [0, \overline{s}_{M-1}]$ would prefer to take the offer, and all sellers with $s \in (\overline{s}_{M-1}]$ would prefer to wait for the subsequent chances of an auction and the final offer. Proceed inductively, we can find a sequence of auctions and sequential offers that implements the optimal interim allocation. In the proof, we show that this special case can be modified to cover all other cases.

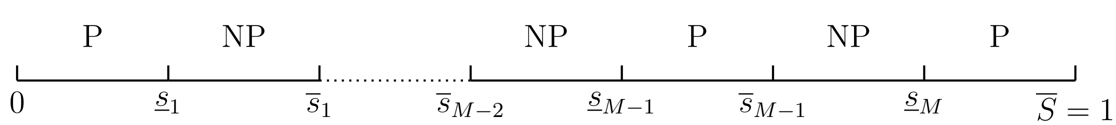

Figure 3: A possible case of the "first two" and the "last four" partition elements where $\overline{S} = 1$. "P" and "NP" means that the partition element is a pooling interval and a non-pooling interval, respectively.

Although the rapid development of technologies significantly reduces the "transaction cost" of holding auctions and making sequential offers,[27] it is worth pointing out that the purpose of Theorem 2 is not to endorse a long sequence of alternating auctions and sequential offers. Instead, Theorem 2 allows us to identify an optimal trading mechanism for some procurement settings with more complicated curvatures caused by the nature of the buyer's preferences or quality distribution. For example, if the valuation function $v(q)$ is S-shaped or inverse S-shaped, then for some quality distributions (say uniform distribution), $g(q)$ can have one peak and one trough.[28] In real-world applications, it is difficult to find circumstances where the virtual surplus g goes up and down very often, so it is unlikely that a "long sequence" is ever needed.

4 Socially optimal procurement mechanisms

In this section, we show that, under certain regularity conditions, a socially optimal procurement mechanism can also be implemented by alternately utilizing sequential offers and auctions. In Section 4.1 we solve problem (6) to get a socially optimal interim allocation, and we discuss how to implement it in Section 4.2. All proofs in this section are relegated to the Online Appendix C.

4.1 Socially optimal interim allocation

To solve problem (6), we set up the Lagrangian with multiplier λ:

$$\mathcal{L} = \int_0^1 \left[\widetilde{h}(s) + \lambda\widetilde{g}(s)\right] \widetilde{P}(s)\,\mathrm{d}s,$$

where $\widetilde{h}(s) = h(F^{-1}(s))$ is the quantile social surplus. We define $\phi(q;\lambda) = h(q) + \lambda g(q)$, and let $\widetilde{\phi}(s;\lambda) = \phi(F^{-1}(s);\lambda)$. Evidently, $\widetilde{\phi}(s;\lambda) = \widetilde{h}(s) + \lambda\widetilde{g}(s)$, which is the quantile "virtual surplus" of the Lagrangian.

We solve the problem by maximizing the Lagrangian over all $\widetilde{P} \in \Omega_w(\widetilde{P}^*)$, and then find an appropriate Lagrangian multiplier such that the complimentary slackness condition holds. More specifically, we iron $\phi(s;\lambda)$ to make sure that the monotonicity constraint holds:

[27] For example, in recent years many procurement processes are completed via online platforms.

[28] By "S-shaped" we mean that the marginal valuation is smaller for low and high qualities, but larger for intermediate qualities, and by interchanging "smaller" and "larger" we get an inverse S-shaped valuation function. The valuation function can be S-shaped if below some quality threshold it is constant in quality (say the quality is "unacceptable" in that case), and strictly concave above the threshold. It can be inverse S shaped if, for instance, we modify the valuation function in Figure 1b so that there is a flat part between the two increasing pieces; this can happen if there is a "gap" between the two markets: intermediate quality input might be too good for the low-end market but not good enough for the high-end market.

let

$$\Phi(s;\lambda) := \int_0^s \widetilde{\phi}(s;\lambda)\,\mathrm{d}x;$$

consequently,

$$\overline{\phi}(s;\lambda) = \frac{\partial}{\partial s}\,\overline{\Phi}(s;\lambda)$$

is the ironed quantile "virtual surplus" of the Lagrangian. Similar to Theorem 1, for all s such that $\overline{\phi}(s;\lambda) > 0$, the socially optimal interim allocation $\hat{P}$ is flat whenever ironing is needed, and coincide with $\widetilde{P}^*(s) = (1-s)^{n-1}$ otherwise. If there exists an interval $[a,b]$ on which $\overline{\phi}(s;\lambda) = 0$, we may need to find some $\overline{P}$ satisfying

$$0 \leq \overline{P} \leq \frac{\int_a^b (1-s)^{n-1}\,\mathrm{d}s}{b-a}$$

and set $\hat{P}(s) = \overline{P}$ on $[a,b]$ to satisfy complementary slackness; the second inequality in the above expression is necessary because Border's condition requires $\hat{P} \prec_w \widetilde{P}^*$.

Theorem 3. *If there exist $\lambda^* \geq 0$, $0 \leq a \leq b \leq 1$ and $b > 0$, and a collection of disjoint intervals $[\underline{s}_i, \bar{s}_i)$ indexed by $i \in \mathcal{I}$, where $[\underline{s}_i, \bar{s}_i) \subseteq [0,a]$, such that*

(i) $a = \sup\left\{s \in [0,1] : \overline{\phi}(s;\lambda^*) > 0\right\}$ *and* $b = \sup\left\{s \in [0,1] : \overline{\phi}(s;\lambda^*) \geq 0\right\}$*;*

(ii) $\overline{\Phi}(s;\lambda^*)$ *is affine on* $[\underline{s}_i, \bar{s}_i)$ *for each* $i \in \mathcal{I}$ *and* $[a,b)$*, and*

(iii) $\overline{\Phi}(s;\lambda^*) = \Phi(s;\lambda^*)$ *on* $[0,a]/\bigcup_{i\in\mathcal{I}}[\underline{s}_i, \bar{s}_i)$*,*

then the interim allocation

$$\hat{P}(s) = \begin{cases} (1-s)^{n-1} & \textit{if } s \in [0,a]/\bigcup_{i\in J}[\underline{s}_i, \overline{s}_i) \\ \frac{\int_{\underline{s}_i}^{\overline{s}_i}(1-s)^{n-1}\mathrm{d}s}{\overline{s}_i - \underline{s}_i} & \textit{if } s \in [\underline{s}_i, \overline{s}_i) \\ \overline{P} & \textit{if } s \in (a,b] \\ 0 & \textit{if } s \in (b,1] \end{cases}$$

satisfying the complementary slackness condition

$$\lambda^* \int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s = 0$$

is optimal.

4.2 Implementation

Observe that if the optimal interim allocation $\hat{P}$ satisfies $a = b$, it can be implemented in a similar manner as the buyer's optimal case we discussed in Section 3.2. If instead $a < b$, what we can do is just implement as before when $s \le a$, and if no potential seller accepts an offer or participates in an auction in these rounds, we know that each of them has quality quantile $s > a$. At this point, the procurement process ends directly without further rounds with probability

$$1 - \frac{\overline{P}}{\int_a^b (1-s)^{n-1}\,\mathrm{d}s/(b-a)},$$

so the buyer does not procure from any of the potential sellers. With complementary probability, another round of sequential offers is tendered, where all sellers with $s \in (a, b]$ would like to accept an offer and others opt out; the procurement process ends after this round. We summarize the discussion above in Theorem 4.

Theorem 4. *Let λ^* be the Lagrangian multiplier associated with the socially optimal interim allocation we identified in Theorem 3. Suppose $\phi(q;\lambda^*) = h(q) + \lambda^* g(q)$ is regular, then a dynamic trading mechanism combining second-price auctions (with a reserve price whenever necessary) and sequential take-it-or-leave-it offers is optimal.*

Corollary 3 and Corollary 4, which are the socially optimal counterparts of Corollary 1 and Corollary 2, identify environments in which $\phi(q;\cdot)$ is strictly decreasing and single-peaked for all $\lambda^* \ge 0$, respectively. Consequently, some simple trading mechanisms are optimal in these cases.

Corollary 3. *Assume the social surplus from trade $h(q) = v(q) - q$ is strictly decreasing, and f is continuously differentiable and log-concave. Then if $\tilde{s} > 0$, one round of second-price auction is socially optimal.*

Corollary 4. *Suppose $v(q)$ is increasing and concave, and F/f is convex in q, then $\phi(q;\lambda^*)$ is single-peaked for all $\lambda^* \ge 0$. Consequently,*

- *if there do not exist $0 < a < b \le 1$ such that $\phi(\cdot;\lambda^*) = 0$ on $[a, b]$, the optimal interim allocation can be implemented either by sequential offers, or by a round of sequential offers followed by an auction;*
- *otherwise, another round of sequential offers might have to be made with positive probability if no one bids in the auction.*

5 Discussion

5.1 The assumption on the sellers' cost

We assume that a seller's cost, or her reservation value, is identical to her quality. While it need not be completely without loss, we believe that it is safe to say that this assumption is innocuous: suppose a seller's cost is given by $c(q)$, where q is her quality, we can redefine the buyer's virtual surplus as $g(q) = v(q) - c(q) - F(q)/f(q)$ and our analysis still goes through; so our main insights do not rely on this assumption. Furthermore, as we will see in Section 5.3, this assumption allows us to conveniently interpret sellers' private information as either quality or cost/reservation value, which expands the applicability of our analysis.

5.2 More general objective functions

The main takeaway that the optimal trading mechanism is a dynamic combination of auction and negotiation continues to hold if the objective is any weighted average of the buyer's payoff and the social surplus. To see this, denote the weight on the buyer's payoff by $\gamma \in [0, 1]$; the optimization problem is

$$\begin{aligned}
\max_{\{p_s\}_{s=1}^n} \quad & \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s - \gamma \frac{F(q_s)}{f(q_s)} \right] p_s(\mathbf{q}) f^n(\mathbf{q}) \, \mathrm{d}\mathbf{q} \\
\text{subject to} \quad & \text{(F)} \\
& P_s(\cdot) \text{ is decreasing for each } s = 1, \ldots, n \\
& \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s - \frac{F\left(q_s\right)}{f\left(q_s\right)} \right] p_s(\mathbf{q}) f^n(\mathbf{q}) \, \mathrm{d}\mathbf{q} \geq 0.
\end{aligned}$$

Letting $h_\gamma(q) := v(q) - q - \gamma[F(q)/f(q)]$, the optimal quantile interim allocation can be found by solving

$$\begin{aligned}
\max_{\widetilde{P} \in \Omega_w(\widetilde{P}^*)} \quad & \int_0^1 h_\gamma(F^{-1}(s)) \widetilde{P}(s) \, \mathrm{d}s \\
\text{s.t.} \quad & \int_0^1 g(F^{-1}(s)) \widetilde{P}(s) \, \mathrm{d}s \geq 0.
\end{aligned}$$

Now proceed like in Section 4.1 and Section 4.2 by simply replacing h by h_γ, we can show that an analog of Theorem 4 holds.

5.3 Stochastic valuation

In the main model, the buyer's valuation is assumed to be a deterministic function of the sellers' quality. In many relevant applications; however, it is natural to assume that the buyer's valuation is a random variable.[29] For concreteness, consider a buyer who would like to contract with one of several potential suppliers to develop a new project, for example, a new weapon or a production line. The cost of supplier s, $c_s \in [0,1]$, is her private information; and the buyer believes that c_s's are independently and identically distributed according to a continuous density function $f_C(\cdot)$. The project's value is not perfectly revealed to the buyer until the end of the development phase at the earliest, which is long after penning the contract. Consequently, at the time of contracting the buyer's valuation is a random variable Ξ. We assume that the realization of Ξ, $\xi \in [\underline{\xi}, \overline{\xi}]$, is not contractable. The buyer believes that Ξ and C are correlated, and that the conditional distribution of Ξ is $f_{\Xi\,|C}(\cdot|c)$. One reasonable assumption about the two random variables Ξ and C can be that they are positively affiliated, or equivalently MTP_2 (Karlin and Rinott, 1980; see also Milgrom and Weber, 1982).

Let $\mathbf{c} = (c_1, \ldots, c_n)$. Given a direct mechanism $\{p_s(\mathbf{c}), t_s(\mathbf{c})\}_{s=1}^n$, where for each cost profile $\mathbf{c}$, $p_s(\mathbf{c})$ specifies the probability that the buyer contracts with supplier s, and $t_s(\mathbf{c})$ is the transfer that the buyer pays to supplier s, the principal's expected payoff can be written as

$$
\begin{aligned}
\pi_b &= \sum_{s=1}^{n} \int_{[0,1]^n} \left[\int_{\underline{\xi}}^{\overline{\xi}} (\xi\, p_s(\mathbf{c}) - t_s(\mathbf{c})) f_{\Xi\,|Q}(\xi|c_s)\, \mathrm{d}\xi \right] f^n(\mathbf{q})\, \mathrm{d}\mathbf{c} \\
&= \sum_{s=1}^{n} \int_{[0,1]^n} \left[\left(\int_{\underline{\xi}}^{\overline{\xi}} \xi f_{\Xi\,|Q}(\xi|c_s)\, \mathrm{d}\xi \right) p_s(\mathbf{c}) - t_s(\mathbf{c}) \right] f^n(\mathbf{c})\, \mathrm{d}\mathbf{c} \\
&= \sum_{s=1}^{n} \int_{[0,1]^n} (\mathbb{E}[\Xi\,|\,Q = c_s]\, p_s(\mathbf{c}) - t_s(\mathbf{c}))\, f^n(\mathbf{c})\, \mathrm{d}\mathbf{c}.
\end{aligned}
$$

If we define $v(c_s) := \mathbb{E}[\Xi\,|\,C = c_s]$, we see from (1) that the problem here is identical to the procurement problem we study above, and the curvature of $v(c_s)$ is governed by the conditional distribution. For example, if Ξ and C are positively affiliated, $v(\cdot)$ is increasing.

Example 3. A manufacturer would like to procure a machine for production. For simplicity, suppose that her valuation is identical to the durability of the machine. She believes that the two potential sellers' costs are identically, independently, and uniformly distributed. Conditional on the cost realization c, her valuation Ξ is distributed according to a Pareto

[29] For conciseness, in Sections 5.3 and 5.4 we only discuss the buyer's optimal design problem; extending the analysis to the socially optimal design problem is straightforward.

distribution with scale 0.5 and shape $2.2 - c$.[30] Consequently,

$$v(c) = \mathbb{E}[\Xi \,|\, C = c] = \frac{1.1 - 0.5c}{1.2 - c},$$

and $g(c) = v(c) - 2c$. We plot g and $\overline{g}$ in Figure 4a. By Theorem 1, the optimal interim allocation is given by

$$\hat{P}(c) = \begin{cases} 1 - c & c < 0.48, \\ 0.26 & c \geq 0.48, \end{cases}$$

which is shown in Figure 4b. Then by Proposition 3, the following trading mechanism is optimal:

- conduct a second-price auction with reserve price $(1 + 0.48)/2 = 0.74$;
- make a take-it-or-leave-it offer of 1 to an arbitrarily selected seller if no bid meets the reserve price. $\lozenge$

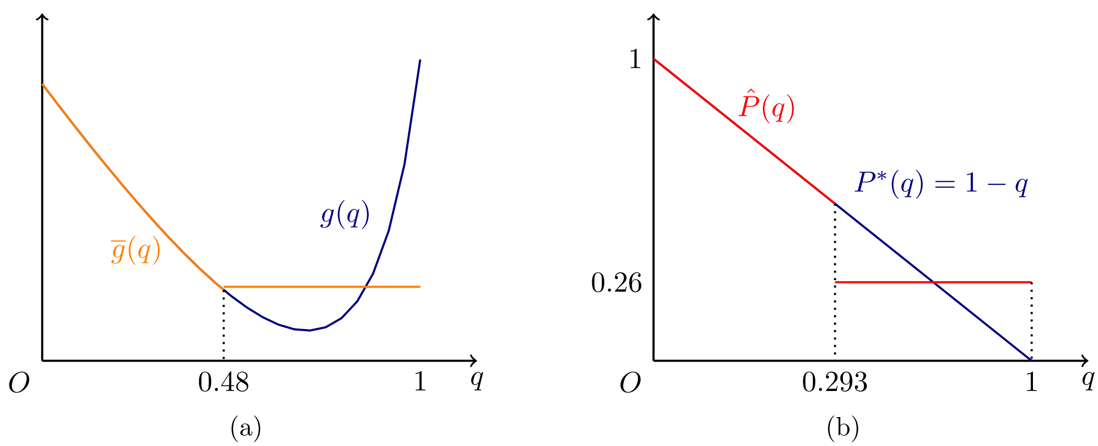

Figure 4: In panel (a), the blue curve is the buyer's quantile virtual surplus $\widetilde{g}$, and the orange curve is the ironed quantile virtual surplus $\overline{g}$; and in panel (b), the blue curve is $P^*(q) = 1 - q$ that appears in Border's condition, and the orange curve is the optimal interim allocation $\hat{P}(q)$.

5.4 Procuring information services

Our framework can also be applied to study the procurement of information services, examples of which include clinical trials and market research. The buyer faces a decision problem under uncertainty: there is a binary payoff-relevant state $\omega \in \Omega = \{0, 1\}$;[31] the buyer's prior

[30] The scale parameter can be interpreted as the length of the machine's warranty.

[31] One can interpret $\omega = 1$ as the "good" state and $\omega = 0$ is the "bad" state.

belief is $\gamma = \mathbb{P}(\omega = 1) \in [0,1]$. She has to take an action $a \in \{S, R\}$, and her payoff is summarized in the matrix

$u(a,\omega)$	S	R
$\omega = 0$	0	θ_0
$\omega = 1$	0	θ_1

where $\theta_0 < 0 < \theta_1$.[32] Without additional information, the buyer's expected payoff is $\overline{V}(\gamma) = \max\{0, \gamma\theta_1 + (1-\gamma)\theta_0\}$.

We model information services by (Blackwell) experiments. An experiment generates two possible signals, $\{\ell, h\}$; one can interpret $s = \ell$ as a recommendation for action S, and $s = h$ is an recommendation for action R. A supplier's quality is her private information; it may summarize, for example, the ability and experiences of its investment team members, (for the clinical research example) the quality of lab facilities, or (for the market research example) the quantity and quality of the data sets it has access to, among many others. The buyer believes that the qualities of the potential suppliers, q_s's are independently and identically distributed, and a supplier with quality q_s offers an experiment

	$s = \ell$	$s = h$
$\omega = 0$	$\kappa_0(q_s)$	$1 - \kappa_0(q_s)$
$\omega = 1$	$1 - \kappa_1(q_s)$	$\kappa_1(q_s)$

where $\kappa_0, \kappa_1 : [0,1] \to [1/2, 1]$.[33] We assume that the buyer's payment cannot be made contingent on the realized signal.

Having access to any experiment from a supplier with quality q_s, the buyer with prior γ observes a signal, updates her beliefs, and then chooses an action; so her expected payoff is

$$V_\gamma(q_s) = \gamma\kappa_1(q_s)\theta_1 + (1-\gamma)(1-\kappa_0(q_s))\theta_0. \tag{13}$$

Consequently, the incremental value of information for her is given by her expected payoff

[32] As is standard in the literature, S and R can be interpreted as "safe" and "risky" action, respectively.

[33] It is without loss of generality to assume that κ_0 and κ_1 maps a quality to a deterministic "success rate": suppose instead that the buyer believes the experiment offered can be written as

	$s = \ell$	$s = h$
$\omega = 0$	α_0	$1 - \alpha_0$
$\omega = 1$	$1 - \alpha_1$	α_1

where α_0 and α_1 are random variables correlated to q_s. Then we just let $\kappa_0(q_s) := \mathbb{E}[\alpha_0 \,|\, Q = q_s]$, and $\kappa_1(q_s) := \mathbb{E}[\alpha_1 \,|\, Q = q_s]$. Assuming that $\kappa_0(q)$ and $\kappa_1(q)$ are greater than or equal to $1/2$ is just to make sure that $s = h$ is relatively more likely to occur under $\omega = 1$ than $\omega = 1$, so it can be interpreted as a recommendation for action R.

$V_\gamma(q_s)$ subtracting the value of prior information, that is,

$$N_\gamma(q_s) = \max\{V_\gamma(q_s) - \overline{V}(\gamma), 0\}. \tag{14}$$

The maximum operator accounts for the case that the buyer's expected payoff from the access to the experiment, $V_\gamma(q_s)$, falls short of the prior value $\overline{V}(\gamma)$, in which case she "ignores" the information provided by the experiment and chooses an action solely based on her prior. Replacing $v(q_s)$ by $N_\gamma(q_s)$, as illustrated by Example 4, our results can be used to analyze this problem.

Example 4. To test an experimental drug, a pharmaceutical company would like to contract with one of three clinical trial service companies to run a treatment trial. The drug can be either "good" ($\omega = 1$) or "bad" ($\omega = 0$),[34] and the pharmaceutical company can choose either to keep developing the drug (action R) or to terminate the development process (action S).

The pharmaceutical company's prior is $\gamma = 2/3$, that is, the drug is good with probability 2/3; and let $\theta_0 = -6$, $\theta_1 = 6$. The clinical trial service companies' qualities are uniformly distributed. For a clinical trial service company with quality q, $\kappa_0(q) = 1/2 + \sqrt{q}/2$, and $\kappa_1(q) = 2/3 + \sqrt{q}/3$. Then $\overline{V}(\gamma) = 2$; so by (14),

$$N_{2/3}(q) = \begin{cases} 0 & q < 1/49, \\ 13\sqrt{q}/5 - 3/5 & q \geq 1/49. \end{cases}$$

Consequently, the virtual surplus is $g(q) = N_{2/3}(q) - 2q$; we plot it as well as its "ironed version" in Figure 5a. We can find the optimal interim allocation by Theorem 1; see Figure 5b for its graph. Then by Proposition 2, it is optimal to sequentially make offers to the three clinical trial companies first, and if none of the offers is accepted, an auction (without a reserve price) is conducted. $\Diamond$

It is useful to compare our setting to a recent paper that also studies "procuring experiments", Yoder (2020). In Yoder (2020), a researcher who is able to undertake experiments plays the role of the only seller or supplier. The researcher's private information, or type, is the marginal cost of a more informative experiment. No matter what the researcher's type is, she is free to choose any experiment,[35] and a more informative experiment is costlier to her. In our framework; however, the lone feasible experiment of the supplier is pinned down by her quality. An interesting possibility is to make the set of feasible experiments for the

[34] For example, a drug is "good" if it reaches some prespecified efficacy and safety levels, and it is "bad" otherwise.

[35] Equivalently, the researcher is allowed to choose any distribution over posteriors whose mean is the prior. See, for example, Kamenica and Gentzkow (2011).

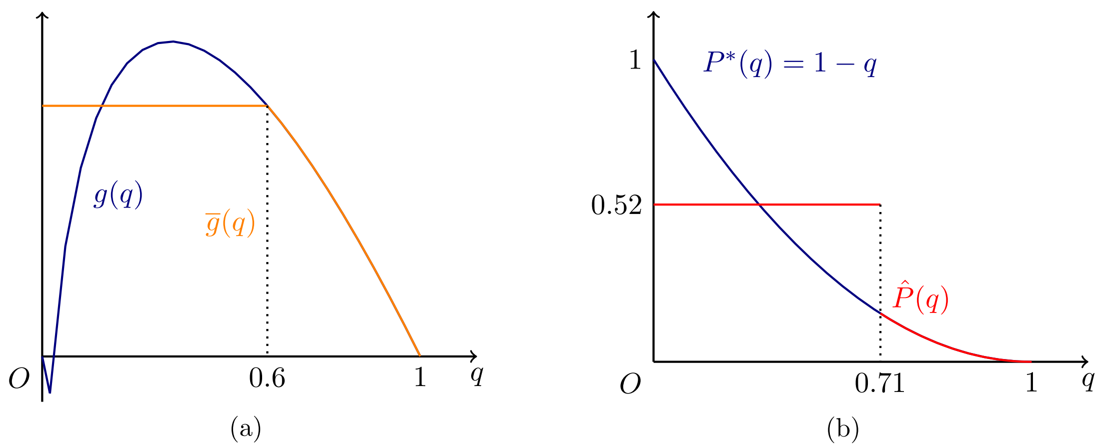

Figure 5: In panel (a), the blue curve is the buyer's virtual surplus $g(q)$, and the orange curve is the ironed virtual surplus $\overline{g}$; and in panel (b), the blue curve is $P^*(q) = (1-q)^2$ that appears in Border's condition, and the red curve is the optimal interim allocation $\hat{P}(q)$.

researcher dependent on her cost or quality, but she is allowed to use any experiment in her feasible set. For example, one can assume that a researcher with a higher cost or quality can conduct more informative experiments; that is, the "informativeness bound" is determined by the researcher's cost or quality.[36] This problem; however, is beyond the scope of this paper and thus left for future research.

6 Conclusion

We explored procurement design problems where the buyer's valuation of the good supplied depends directly on its quality, and the quality is both unverifiable and unobservable. We analyzed both buyer's payoff maximization and social surplus maximization problems. To obtain the main results, we employed a reduced form approach, based on techniques on linear optimization under a majorization constraint. We found that in each of the problems, the two commonly used procurement methods, namely auction and negotiation, are suboptimal unless the virtual valuation is monotone. However, the optimal mechanisms can be implemented by a dynamic combination of them.

An interesting question for future research is under what conditions either auction or negotiation is "approximately optimal" in the sense that it captures a substantial portion of the surplus achieved by the optimal mechanism. In such cases, one of the two common procurement methods might be favored because of its simplicity.

[36] This assumption is reasonable in many applications. For example, a credit rating agency that makes more precise predictions usually hires more experienced investigators, which leads to higher costs.

Appendices

A Results on majorization

Denote the set of decreasing functions in $L^1(0,1)$ that are majorized by f by

$$\Omega(f) := \{g \in L^1(0,1) : g \text{ is decreasing, } g \prec f\};$$

similarly, denote the "weak majorization set" by

$$\Omega_w(f) := \{g \in L^1(0,1) : g \text{ is decreasing, } g \prec_w f\}.$$

The following results are taken from Kleiner et al. (2020) and modified to our environment. Let A be an arbitrary subset of a topological vector space, we denote the set of its extreme points by $\mathrm{ext}A$.

Theorem 5 (Theorem 1 in Kleiner et al., 2020)**.** *Let $f \in L^1(0,1)$ be decreasing. Then $h \in \mathrm{ext}\Omega(f)$ if and only if there exists a collection of disjoint intervals $[\underline{x}_i, \overline{x}_i)$ indexed by $i \in I$ such that for almost all $x \in [0,1]$,*

$$h(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{i\in I} [\underline{x}_i, \overline{x}_i) \\ \frac{\int_{\underline{x}_i}^{\overline{x}_i} f(s)\mathrm{d}s}{\overline{x}_i - \underline{x}_i} & \text{if } x \in [\underline{x}_i, \overline{x}_i)\,. \end{cases}$$

For $B \subseteq [0,1]$, denote by $\mathbf{1}_B(x)$ the indicator function of B: it equals 1 if $x \in B$ and 0 otherwise.

Corollary 5 (Corollary 2 in Kleiner et al., 2020)**.** *Let $f \in L^1(0,1)$ be decreasing. Then $h \in \mathrm{ext}\Omega_w(f)$ if and only if there exists $\theta \in [0,1]$ such that $h \in \mathrm{ext}\Omega(f \cdot \mathbf{1}_{[0,\theta]})$ and $h(x) = 0$ for almost all $x \in (\theta, 1]$.*

Now consider the problem

$$\max_{m\in\Omega(f)} \int_0^1 c(x)r(x)\,\mathrm{d}x, \tag{15}$$

where $f \in L^1(0,1)$ is strictly decreasing, and c is a bounded function. Define

$$C(x) = \int_0^x c(s)\,\mathrm{d}s,$$

and let $\overline{C}$ be its concave hull. Proposition 4 characterizes a solution to problem (15).

Proposition 4 (Proposition 2 in Kleiner et al., 2020). *Let $h \in \text{ext}\Omega(f)$, and let $\{[\underline{x}_i, \bar{x}_i) : i \in I\}$ be the collection of intervals described in Theorem 5. Then h is optimal if and only if $\overline{C}$ is affine on $[\underline{x}_i, \bar{x}_i)$ for each $i \in I$ and $\overline{C} = C$ otherwise.*

B Proofs for Section 3

B.1 Proof of Theorem 1

Because the objective function of problem (5) is linear, by Bauer's maximum principle (Aliprantis and Border (2006), Theorem 7.69, page 298), the maximum is attained at an extreme point $\hat{P}$ of $\Omega_w(\widetilde{P}^*)$. By Corollary 5, there exists $\bar{s} \in [0,1]$ such that $\hat{P}$ is an extreme point of $\Omega(\widetilde{P}^* \cdot \mathbf{1}_{[0,\bar{s}]})$ and equals zero on $[\bar{s}, 1]$. Furthermore, the optimality of $\hat{P}$ requires that the type $\bar{q} = F^{-1}(\bar{s})$ must satisfy $\overline{g}(\bar{q}) = 0$; so setting $\bar{s} = \sup\{s \in [0,1] : \overline{g}(s) \geq 0\} = \overline{S}$ suffices. Then Theorem 5 implies that $\hat{P}$ must take the form of

$$\hat{P}(s) = \begin{cases} (1-s)^{n-1} & \text{if } s \in [0, \overline{S}] / \bigcup_{i \in \mathcal{I}} [\underline{s}_i, \overline{s}_i)\,, \\ \frac{\int_{\underline{s}_i}^{\overline{s}_i} (1-s)^{n-1} \mathrm{d}s}{\overline{s}_i - \underline{s}_i} & \text{if } s \in [\underline{s}_i, \overline{s}_i)\,, \\ 0 & \text{if } s \in (\overline{S}, 1]; \end{cases}$$

and by Proposition 4, the collection $\{[\underline{s}_i, \overline{s}_i) \subseteq [0, \overline{S}] : i \in \mathcal{I}\}$ is pinned down by the intervals that $\overline{G}$ is affine on.

B.2 Construction of prices and equilibrium in the proof of Proposition 2

In what follows we construct $p_1, p_2, \ldots, p_n$ such that the following strategy is an equilibrium of this game: sellers with valuations $q \leq F^{-1}(\hat{s})$ accept the take-it-or-leave-it offer whenever it is tendered to her; and if an auction is conducted, all sellers submit a bid equals to her valuation.

To simplify notation, we let $\hat{q} = F^{-1}(\hat{s})$; and for brevity of the proof, we assume $\overline{S} = 1$: to account for other cases, we only need to set a reserve price $F^{-1}(\overline{S})$ for the auction. Now suppose all other potential sellers are expected to follow the strategy described above. For each type q seller, if she enters the final auction, her expected revenue is

$$M(q) = \begin{cases} \mathbb{E}\left[q^{(n-1)} \;\middle|\; q^{(n-1)} > \hat{q}\right] - q & \text{if } q < \hat{q}, \\ \mathbb{P}\left(q^{(n-1)} > q \;\middle|\, q^{(n-1)} > \hat{q}\right) \left(\mathbb{E}\left[q^{(n-1)} \;\middle|\; q^{(n-1)} > q\right] - q\right) & \text{if } q \geq \hat{q}; \end{cases}$$

where $q^{(n-1)}$ is the lowest of her $n-1$ opponents' types. By definition, $M(q)+q$ is constant on $[0,\hat{q}]$; and for $q>\hat{q}$,

$$M(q)+q=\frac{(1-F(q))^{n-1}}{(1-\hat{s})^{n-1}}\left[\frac{\int_q^1 x\,\mathrm{d}K(x)}{(1-F(q))^{n-1}}-q\right]+q=\frac{\int_q^1 x\,\mathrm{d}K(x)}{(1-\hat{s})^{n-1}}+\left(1-\frac{(1-F(q))^{n-1}}{(1-\hat{s})^{n-1}}\right)q,$$

where K is the distribution of $q^{(n-1)}$: $K(x)=1-(1-F(x))^{n-1}$. So for $q>\hat{q}$,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}q}(M(q)+q)&=-\frac{q(n-1)f(q)(1-F(q))^{n-2}}{(1-\hat{s})^{n-1}}+\left(1-\frac{(1-F(q))^{n-1}}{(1-\hat{s})^{n-1}}\right)+\frac{q(n-1)f(q)(1-F(q))^{n-2}}{(1-\hat{s})^{n-1}}\\&=\left(1-\frac{(1-F(q))^{n-1}}{(1-\hat{s})^{n-1}}\right)>0,\end{aligned}$$

which implies that $M(q)+q$ is strictly increasing on $(\hat{q},1]$.

Consider Seller n first. She knows that if she rejects the offer p_n, the auction occurs with probability 1; so we let

$$p_n-\hat{q}=M(\hat{q});$$

the properties of $M(q)+q$ we discussed above imply that $q\leq\hat{q}$ would accept the offer and wait for the auction otherwise. For Seller $n-1$, she knows that if she rejects the offer p_n, the auction occurs if and only if Seller n's type is greater than $\hat{q}$, which happens with probability $1-\hat{s}$; so for type $\hat{q}$ Seller $n-1$ to be indifferent, we set

$$p_{n-1}-\hat{q}=(1-\hat{s})M(\hat{q}).$$

Following the reasoning above, for $j=1,\ldots,n-2$, Seller j's take-it-or-leave-it offer is

$$p_j=F^{-1}(\hat{s})+(1-\hat{s})^{n-j}M(\hat{q}).$$

By construction, $p_1<p_2<\cdots<p_{n-1}<p_n$. Consequently, a seller with type $q<F^{-1}(\hat{s})$ accepts the take-or-leave-it offer once it is tendered to her, and otherwise waits for the chance of an auction.

B.3 Proof of Proposition 3 continued

Suppose $\overline{S}\geq\tilde{s}>0$, so $\overline{S}=1$ and (12) is the optimal interim allocation. Because $\overline{S}=1$, we set the reserve price R such that type $\widetilde{q}=F^{-1}(\tilde{s})$ seller is indifferent between bidding in the auction and wait until the chance of being selected to get the offer of 1; more precisely, we set

$$R = \frac{n-1}{n}\widetilde{q} + \frac{1}{n} < 1.$$

We claim that the following strategy is an equilibrium of this game: in the auction phase, sellers with valuations $q \leq \widetilde{q}$ submit bids equal to their valuations; in the offer phase, all sellers accept the take-it-or-leave-it offer of 1 when it is tendered to her. To see this, suppose all other sellers follow this strategy. Then when a seller's true quality is q, by bidding $q' \leq \widetilde{q}$, the expected revenue is

$$\int_{q'}^{\widetilde{q}} (x-q)\,\mathrm{d}K(x) + (1-K(\widetilde{q}))\,(R-q) = \int_{q'}^{\widetilde{q}} (x-q)\,\mathrm{d}K(x) + (1-\tilde{s})^{n-1}\,(R-q)\,,$$

where the second equality follows from the definition of K. If $q \leq \widetilde{q}$, the integral term above is positive and achieves its maximum at $q' = q$. By bidding $q' \in [\widetilde{q}, R]$, the seller wins only if $q^{(n-1)} \geq \widetilde{q}$; in this case her expected revenue is

$$(1-\tilde{s})^{n-1}\,(R-q)\,;$$

and bidding $q' > R$ makes the seller never win. Thus, if a seller's quality $q \leq \widetilde{q}$, it is optimal for her to bid her true quality.

Observe that any seller with type $q > R$ would prefer to wait for the chance of a take-it-or-leave-it offer. For $q \in [\widetilde{q}, R]$, if she submits a bid in the auction, her expected payoff is maximized at bidding $q' = \widetilde{q}$, which yields

$$(1-\tilde{s})^{n-1}\,(R-q)\,;$$

and if she does not, her expected payoff from the offer is

$$\frac{1}{n}(1-\tilde{s})^{n-1}(1-q),$$

where $(1-\tilde{s})^{n-1}$ is the probability that the game enters the offering phase, and $1/n$ is the probability that an offer is tendered to her; if she receives an offer, she accepts it for sure, which yields a payoff of $1-q$. By definition of R, any seller with type $q \leq \widetilde{q}$ prefers to submit a bid equal to her true quality in the auction, and a seller with $q > \widetilde{q}$ would prefer to wait for the chance of a take-it-or-leave-it offer.

Consequently, all types with $q < F^{-1}(\tilde{s})$ make a bid in the auction, and otherwise wait for a potential offer. It is then not difficult to see that the interim allocation of the trading mechanism, in this case, is exactly (12): in particular, on $[\tilde{s}, 1]$,

$$\frac{\int_{\tilde{s}}^{1}(1-s)^{n-1}\mathrm{d}s}{1-\tilde{s}} = \frac{(1-\tilde{s})^{n-1}}{n},$$

which is exactly the probability that a seller with quality quantile $s \in [\tilde{s}, 1]$ receives a take-it-or-leave-it offer (and this seller accepts with probability one).

B.4 Proof of Theorem 2

Like in the proof of Proposition 2, to simplify exposition, we let $\overline{S} = 1$. The proof proceeds as follows: we first design a sequence of sequential take-it-or-leave-it offers and second-price auctions with reserve prices that induces an equilibriumof the game played by the sellers, such that all seller types whose quantile falls in a pooling interval would accept some take-it-or-leave-it offer that is designed for sellers in that interval, and all seller types whose quantile falls in a non-pooling interval would submit a bid in a second-price auction that is designed for sellers in that interval; then we show that the interim allocation rule induced by this equilibrium coincides with $\hat{P}$.

In what follows we prove the theorem for a special case, and after that we show that focusing on this case is in fact without loss of generality. We assume that

(1) pooling intervals and non-pooling intervals are alternating, that is, except for the "last" interval, a pooling interval is always followed by a non-pooling interval, and a non-pooling interval is always followed by a pooling interval;

(2) the first interval is a non-pooling interval, and the last interval is a pooling interval.

Since there are M pooling intervals, in this case, there are also M non-pooling intervals. We claim that the following mechanism, with appropriately chosen offers and reserve prices, is optimal:

- *auction round 1:* conduct a second-price auction with reserve price $R^{(1)} > 0$;
- *sequential offer round 1:* randomly number the sellers from 1 to n, and sequentially tender take-it-or-leave-it offer $p_j^{(1)}$ to Seller j, $j = 1, \ldots, n$, where $p_1^{(1)} < p_2^{(1)} < \cdots < p_n^{(1)}$;
- *auction round 2:* conduct a second-price auction with reserve price $R^{(2)}$, where $R^{(1)} < R^{(2)}$;

- *sequential offer round 2:* randomly number the sellers from 1 to n, and sequentially tender take-it-or-leave-it offer $p_j^{(2)}$ to Seller j, $j = 1, \ldots, n$, where $p_1^{(2)} < p_2^{(2)} < \cdots < p_n^{(2)}$;

- $\ldots$;

- *auction round M:* conduct a second-price auction with reserve price $R^{(M)}$, where $R^{(M-2)} < R^{(M-1)} < 1$;

- *the final round:* choose a seller at random and tender a take-it-or-leave-it offer of 1 to her.

To prove this, we first show that an equilibrium of the game induced by the mechanism we described above consists of the following strategies: for $j = 1, \ldots, M-1$, sellers whose quantiles are in j-th pooling interval only accept an offer from sequential offer round j, and sellers whose quantiles are in k-th non-pooling interval only make bids equal to their qualities in auction round k, $k = 1, \ldots, M$; all sellers whose quantiles fall in the last pooling interval only accept the final take-it-or-leave-it offer of 1.

To see this, suppose all other sellers follow the aforementioned strategies; we choose $R^{(M)}$ like in the proof of Proposition 3, then if a seller's quality quantile falls in the last pooling interval $[\underline{s}_M, 1]$,[37] she prefers to wait for the chance of the final offer than to make a bid in the M-th auction, and all other types prefer the M-th auction to the final offer. Next, we set the n take-it-or-leave-it offers $p_1^{(M-1)}, p_2^{(M-1)}, \ldots, p_n^{(M-1)}$ in a similar manner as in the proof of Proposition 2; so if a seller's quality quantile is in the $M-1$-th non-pooling interval $[\overline{s}_{M-1}, \underline{s}_M)$, she prefers to wait for the possibilities of an auction and the final offer than to accept an offer in the $M-1$-th round of sequential offers, and if $s \in [0, \overline{s}_{M-1})$ she would prefer the opposite. Then we let

$$R^{(M-1)} = F^{-1}(\underline{s}_{M-1}) + \frac{1}{n} \sum_{k=1}^{n} \left(\frac{1 - \overline{s}_{M-1}}{1 - \underline{s}_{M-1}} \right)^{k-1} \left(p_k^{(M-1)} - F^{-1}(\underline{s}_{M-1}) \right).$$

An argument analogous to the proof of Proposition 3 shows that for each seller with $s \in [0, \underline{s}_{M-1})$, it is optimal for her to bid her quality in the auction. Now consider a seller whose quality quantile are in the $M-1$-th pooling interval, that is, $s \in [\underline{s}_{M-1}, \overline{s}_{M-1})$: if she submits a bid in the $M-1$-th auction, her expected payoff is maximized by bidding $F^{-1}(\underline{s}_{M-1})$, which yields

$$\left(\frac{1 - \underline{s}_{M-1}}{1 - \overline{s}_{M-2}} \right)^{n-1} \left(R^{(M-1)} - F^{-1}(s) \right),$$

[37] Note that in this case, we have $\overline{s}_M = 1$.

where the first term is the probability that the buyer procures from her conditional on that the $M-1$-th auction is conducted, and the second term is the payoff when she wins; and if she waits for the chance of getting into sequential offer round $M-1$,[38] her expected payoff is

$$\left(\frac{1-\underline{s}_{M-1}}{1-\overline{s}_{M-2}}\right)^{n-1}\left[\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1-\overline{s}_{M-1}}{1-\underline{s}_{M-1}}\right)^{k-1}\left(p_k^{(M-1)}-F^{-1}(s)\right)\right]. \tag{16}$$

To understand (16), note that the term outside of the squared brackets is the probability that no one wins the auction conditional on that the $M-1$-th auction is conducted, and the squared bracket is her expected payoff in the $M-1$-th round of sequential offers: if she is labeled as Seller k, when an offer is tendered to her she gets $p_k^{(M-1)}-F^{-1}(s)$, which happens only if all $k-1$ sellers selected before her have quality quantile above $\overline{s}_{M-1}$. Define

$$\Gamma_{(M-1)}(q)=q+\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1-\overline{s}_{M-1}}{1-\underline{s}_{M-1}}\right)^{k-1}\left(p_k^{(M-1)}-q\right);$$

$\Gamma_{(M-1)}$ is strictly increasing whenever $\overline{s}_{M-1}>\underline{s}_{M-1}$ since

$$\begin{aligned}\Gamma'_{(M-1)}(q)&=1-\frac{1}{n}\sum_{k=1}^{n}\left(\frac{1-\overline{s}_{M-1}}{1-\underline{s}_{M-1}}\right)^{k-1}\\&\geq 1-\frac{n}{n}\left(\frac{1-\overline{s}_{M-1}}{1-\underline{s}_{M-1}}\right)\\&=\left(\frac{\overline{s}_{M-1}-\underline{s}_{M-1}}{1-\underline{s}_{M-1}}\right)>0;\end{aligned}$$

then by definition of $R^{(M-1)}$, we see that all sellers with $s\in[0,\underline{s}_{M-1}]$ prefer the $M-1$-th auction to the subsequent offers, and all sellers with $s\in[\underline{s}_{M-1},\overline{s}_{M-1})$ prefer the opposite. Proceed inductively, we can show that the proposed strategies indeed constitute an equilibrium.

Therefore, if a seller's quality quantile s falls in a non-pooling interval, the only way that the buyer procures from her is through a second-price auction, which implies that the corresponding interim allocation she receives is $P(s)=(1-s)^{n-1}$: this is exactly the probability that all other $n-1$ sellers have higher quality. If her quality quantile s falls in a pooling interval, say $s\in[\underline{s}_j,\overline{s}_j)$ for some $j\in\{1,\ldots,M\}$, then the only way that the buyer procures from her is through a take-it-or-leave-it offer, which implies that the corresponding

[38]Note that the seller would not want to wait even longer since we have shown that any $s\in[0,\overline{s}_{M-1})$ prefers accepting an offer in sequential offer round $M-1$ to waiting for any subsequent chances.

interim allocation she receives is

$$P(s) = \frac{\sum_{k=0}^{n-1}(1-\underline{s}_j)^{n-1-k}(1-\overline{s}_j)^k}{n}. \tag{17}$$

To understand (17), observe that if the seller is chosen as the $k+1$-th seller, to make sure that she sells the good to the buyer, we need all of the k sellers who get the offers before her have quality higher than $F^{-1}(\overline{s}_j)$ (so that none of them accept the offer tendered to them in this round) and the other $n-1-k$ sellers' qualities are above $F^{-1}(\underline{s}_j)$ (so the procurement mechanism does not stop before this round). Then by the algebraic identity

$$a^n - b^n = (a-b)\sum_{k=1}^{n} a^{k-1}b^{n-1-k},$$

letting $a = (1-\overline{s}_j)$ and $b = 1-\underline{s}_j$ we have

$$\frac{\sum_{k=0}^{n-1}(1-\underline{s}_j)^{n-1-k}(1-\overline{s}_j)^k}{n} = \frac{(1-\overline{s}_j)^n - (1-\underline{s}_j)^n}{n(\overline{s}_j - \underline{s}_j)};$$

but by the fundamental theorem of calculus, the right-hand side is exactly

$$\frac{\int_{\underline{s}_j}^{\overline{s}_j}(1-s)^{n-1}\mathrm{d}s}{\overline{s}_j - \underline{s}_j}.$$

Therefore, the equilibrium we proposed is consistent with the optimal interim allocation rule identified in Theorem 1.

Now we argue that every other case can be seen as a simple variant of the special case we analyzed. It is possible to have two consecutive pooling intervals, but based on our proof below, we can always put a "vacuous auction" in between: the reserve price of that auction is set to make sure that only the boundary point of the two consecutive pooling intervals is indifferent between the first offer, the auction, and the second offer, and all other types prefer an offer (either the first one or the second). Moreover, if the "first" interval is a pooling interval, we can set the reserve price of the first auction round to be 0; and if the "last" interval is a non-pooling interval, we can just set the reserve price of the last round of auction to be 1.

Therefore, we conclude that whenever the virtual surplus has finitely many peaks and troughs, any buyer's optimal procurement mechanism can be implemented by a dynamic trading scheme that alternates between take-it-or-leave-it offers and second-price auctions with reserve prices.

References

ALIPRANTIS, C. D. AND K. C. BORDER (2006): *Infinite Dimensional Analysis: A Hitchhiker's Guide*, Berlin: Springer, 3rd ed.

BAGNOLI, M. AND T. BERGSTROM (2005): "Log-Concave Probability and Its Applications," *Economic Theory*, 26, 445–469.

BONACCORSI, A., T. P. LYON, F. PAMMOLLI, AND G. TURCHETTI (2000): "Auctions vs. Bargaining: An Empirical Analysis of Medical Device Procurement," Indiana University working paper.

BORDER, K. C. (1991): "Implementation of Reduced Form Auctions: A Geometric Approach," *Econometrica*, 59, 1175–1187.

——— (2007): "Reduced Form Auctions Revisited," *Economic Theory*, 31, 167–181.

CHE, Y.-K. (2008): "Procurement," in *The New Palgrave Dictionary of Economics*, ed. by L. Blume and S. Durlauf, London: Palgrave Macmillan, 2nd ed.

CHE, Y.-K., D. CONDORELLI, AND J. KIM (2018): "Weak Cartels and Collusion-Proof Auctions," *Journal of Economic Theory*, 178, 398–435.

DECAROLIS, F. AND C. GIORGIANTONIO (2015): "Local Public Procurement Regulations: The Case of Italy," *International Review of Law and Economics*, 43, 209–226.

GERSHKOV, A., J. K. GOEREE, A. KUSHNIR, B. MOLDOVANU, AND X. SHI (2013): "On the Equivalence of Bayesian and Dominant Strategy Implementation," *Econometrica*, 81, 197–220.

GERSHKOV, A., B. MOLDOVANU, P. STRACK, AND M. ZHANG (2021): "A Theory of Auctions with Endogenous Valuations," *Journal of Political Economy*, 129, 1011–1051.

HART, S. AND P. J. RENY (2015): "Implementation of Reduced Form Mechanisms: A Simple Approach And A New Characterization," *Economic Theory Bulletin*, 3, 1–8.

KAMENICA, E. AND M. GENTZKOW (2011): "Bayesian Persuasion," *American Economic Review*, 101 (6), 2590–2615.

KARLIN, S. AND Y. RINOTT (1980): "Classes of orderings of measures and related correlation inequalities. I. Multivariate totally positive distributions," *Journal of Multivariate Analysis*, 10, 467–498.

Kleiner, A., B. Moldovanu, and P. Strack (2020): "Extreme Points and Majorization: Economic Applications," Arizona State University working paper.

Lalive, R., A. Schmutzler, and C. Zulehner (2015): "Auctions vs Negotiations in Public Procurement: Which Works Better?" University of Lausanne working paper.

Lopomo, G., N. Persico, and A. T. Villa (2022): "Optimal Procurement with Quality Concerns," Duke University Working Paper.

Luenberger, D. G. (1969): *Optimization by Vector Space Methods*, New York: John Wiley and Sons.

Manelli, A. M. and D. R. Vincent (1995): "Optimal Procurement Mechanisms," *Econometrica*, 63, 591.

——— (2004): "Duality in Procurement Design," *Journal of Mathematical Economics*, 40, 411–428.

Maskin, E. and J. Riley (1984): "Optimal Auctions with Risk Averse Buyers," *Econometrica*, 52, 1473–1518.

Matthews, S. A. (1984): "On the Implementability of Reduced Form Auctions," *Econometrica*, 52, 1519–1522.

Mierendorff, K. (2011): "Asymmetric Reduced Form Auctions," *Economics Letters*, 110, 41–44.

Milgrom, P. R. and R. J. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089–1122.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operations Research*, 6 (1), 58–73.

Toikka, J. (2011): "Ironing without Control," *Journal of Economic Theory*, 146, 2510–2526.

Yoder, N. (2020): "Designing Incentives for Heterogeneous Researchers," University of Georgia working paper.

Zhang, M. (2018): "Auctions vs. Negotiations: Optimal Selling Mechanism with Endogenous Bidder Values," University of Bonn working paper.

Online Appendix

C Proofs for Section 4

C.1 Proof of Theorem 3

Let
$$H(s) = \int_0^s \widetilde{h}(x)\,\mathrm{d}x \quad \text{and} \quad \Phi(s;\lambda) = \int_0^s \widetilde{\phi}(s;\lambda)\,\mathrm{d}x,$$
and let $\overline{H}$ and $\overline{\Phi}$ be the concave hulls of H and Φ, respectively. We further define $\{[\underline{x}_i, \overline{x}_i)\}_{i\in I}$ and $\{[\underline{y}_i, \overline{y}_i)\}_{i\in J}$ be the collections of intervals on which $\overline{H}$ and $\overline{\Phi}$ are affine, respectively. Now let
$$\overline{h}(s) = \overline{H}'(s), \quad \text{and} \quad \overline{\phi}(s;\lambda) = \frac{\partial}{\partial s}\overline{\Phi}(s;\lambda),$$
so $\overline{h}(s)$ and $\overline{\phi}(s;\lambda)$ are the ironed quantile social surplus and the ironed quantile "virtual surplus" of the Lagrangian, respectively. By construction, both of them are decreasing in s. Define
$$S_0 = \left\{s \in [0,1] : \overline{h}(s) = 0\right\},$$
so S_0 is the set of points on which the ironed quantile social surplus is zero. There are two cases that S_0 is empty: either $\overline{h}(1) > 0$, or $\overline{h}(0) < 0$. Because $\overline{h}(0) < 0$ represents the uninteresting case that it is socially undesirable to trade under incomplete information, so we assume that if S_0 is empty, we have $\overline{h}(1) > 0$.

By Corollary 1 on page 219 and Theorem 2 on page 221 of Luenberger (1969), $\hat{P} \in \Omega_w(\widetilde{P}^*)$ solves problem (6) if and only if there exists $\lambda \geq 0$ such that $\hat{P}$ maximizes $\mathcal{L}$, and the complementary slackness condition
$$\lambda \int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s = 0$$
holds. Consequently, an optimal $\hat{P}$ can be found using the following algorithm:

Step 1. Check that if there exists $\hat{s} \in [0,1]$, either $\hat{s} \in S_0$, or $S_0 = \varnothing$ and $\hat{s} = 1$ such that
$$\hat{P}(s) = \begin{cases} (1-s)^n & \text{if } s \in [0,\hat{s}]/\bigcup_{i\in I}[\underline{x}_i, \overline{x}_i) \\ \frac{\int_{\underline{x}_i}^{\overline{x}_i}(1-s)^{n-1}\mathrm{d}s}{\overline{x}_i - \underline{x}_i} & \text{if } s \in [\underline{x}_i, \overline{x}_i) \\ 0 & \text{if } s \in (\hat{s}, 1] \end{cases}$$

satisfies

$$\int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s \geq 0.$$

If so, we can set $\lambda = 0$, which implies that the quantile "virtual surplus" of the Lagrangian coincides with the ironed quantile social surplus: $\overline{\phi}(s;0) = \overline{h}(s)$; and $\hat{P}$ solves problem (6). If not, go to **Step 2**.

Step 2. We must have $\lambda > 0$, otherwise we could have found an $\hat{s}$ in **Step 1**. Now we search for $\lambda > 0$ such that there exists unique $\tilde{s}$ that $\overline{\phi}(s;\lambda) = 0$, and the "induced interim allocation"

$$\hat{P}(s) = \begin{cases} (1-s)^n & \text{if } s \in [0,\tilde{s}]/\bigcup_{i\in J}[\underline{y}_i, \overline{y}_i) \\ \frac{\int_{\underline{y}_i}^{\overline{y}_i}(1-s)^{n-1}\mathrm{d}s}{\overline{y}_i - \underline{y}_i} & \text{if } s \in [\underline{y}_i, \overline{y}_i) \\ 0 & \text{if } s \in (\tilde{s}, 1] \end{cases}$$

satisfies

$$\int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s = 0.$$

If we can find such $(\lambda, \tilde{s})$ pair, $\hat{P}$ solves problem (6); if not, go to **Step 3**.

Step 3. Then there must exist an interval $[a,b] \subseteq [0,1]$, where $a < b$, such that $\overline{\phi}(\cdot;\lambda) = 0$ on $[a,b]$, and there exists $\overline{P}$ with

$$0 \leq \overline{P} \leq \frac{\int_a^b (1-s)^{n-1}\,\mathrm{d}s}{b-a}$$

such that

$$\hat{P}(s) = \begin{cases} (1-s)^n & \text{if } s \in [0,a]/\bigcup_{i\in J}[\underline{y}_i, \overline{y}_i) \\ \frac{\int_{\underline{y}_i}^{\overline{y}_i}(1-s)^{n-1}\mathrm{d}s}{\overline{y}_i - \underline{y}_i} & \text{if } s \in [\underline{y}_i, \overline{y}_i) \\ \overline{P} & \text{if } s \in (a,b] \\ 0 & \text{if } s \in (b, 1] \end{cases}$$

satisfies

$$\int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s = 0$$

solves problem (6).

C.2 Proof of Corollary 3

Because f is continuously differentiable and log-concave, $-F/f$ is decreasing. Then since h is strictly decreasing, $g(q) = h(q) - F(q)/f(q)$ is also strictly decreasing, and so are $\widetilde{g}$

and $\widetilde{h}$. Therefore, the "virtual surplus" of the Lagrangian, $\widetilde{\phi}(s;\lambda)=\widetilde{h}(s)+\lambda\widetilde{g}(s)$ must be strictly decreasing for any $\lambda \geq 0$. Consequently, there must exist unique $\check{s} \in [0,1]$ such that $\widetilde{\phi}(\check{s};\lambda)=0$. The result thus follows.

C.3 Proof of Corollary 4

If $\overline{\phi}(0;\lambda^*) < 0$, then it is socially optimal not to procure from any potential seller. If $\overline{\phi}(s;\lambda^*)=0$ on an interval $[0,b]$ with $b>0$, then following Step 3 in the algorithm in the proof of Theorem 3, it is optimal to make one round offer with probability p, where p solves

$$\int_0^1 \widetilde{g}(s)\widetilde{P}(s)\,\mathrm{d}s = \int_0^b \widetilde{g}(s)p\frac{\int_0^b (1-t)^{n-1}\,\mathrm{d}t}{b}\,\mathrm{d}s = 0.$$

Note that the second equality above holds if and only if $\int_0^b \widetilde{g}(s)\,\mathrm{d}s = 0$, so we can set $p=1$. Therefore, in this case, we only need one round of sequential offers.

Now assume $\overline{\phi}(0;\lambda^*)=m>0$. Because both $v(q)-q$ and $-F(q)/f(q)$ are concave under our assumptions, so their weighted sum $\phi(q;\lambda)$ is also concave and a fortiori single-peaked. Consequently, there must exist $c \in [0,1]$ such that $\overline{\phi}(s;\lambda^*)=m$ on $[0,c)$ and decreasing on $[c,1]$. Then by Theorem 4, if there exists unique $\check{s} \in [c,1]$ such that $\widetilde{\phi}(\check{s};\lambda^*)=0$, a round of sequential offers followed by an auction is optimal; otherwise another round of offers might be rendered with positive probability.

D Single-peakedness of buyer's virtual surplus

In this section, we identify a set of conditions on primitives that ensures the buyer's virtual surplus $g(q)$ is single-peaked.

Lemma 3. *Assume that both $v(q)$ and $f(q)$ are twice continuously differentiable, and $v'(q) \neq 0$ for all $q \in [0,1]$, then g is single-peaked if and only if for all $q^* \in L_g = \{q \in [0,1] \mid g'(q) = 0\}$, we have*

$$-\frac{v''(q^*)}{v'(q^*)} \geq -\frac{\mathrm{d}}{\mathrm{d}q}\log\left(1+\frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{F(q^*)}{f(q^*)}\right)\right). \tag{18}$$

Note that the left-hand side of (18) can be interpreted as the Arrow-Pratt coefficient of absolute risk aversion of buyer's valuation function v evaluated at q^*. Loosely speaking, for (18) to hold, v has to be risk averse enough at all $q^* \in L_g$, or (even looser) v is "not too convex" around q^*'s. Whether it is "enough" or "not too convex" is determined by the quality distribution.

Proof. Observe that $g(q)$ is single-peaked if and only if for any $q^* \in [0,1]$ such that $g'(q^*) = 0$, we have $g''(q^*) \leq 0$.[39] Now

$$g'(q^*) = 0 \Leftrightarrow v'(q^*) = 1 + \frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{F(q^*)}{f(q^*)}\right).$$

But then $g''(q^*) = v''(q^*) - [F(q^*)/f(q^*)]''$ has the same sign as

$$\frac{v''(q^*)}{v'(q^*)} - \frac{[F(q^*)/f(q^*)]''}{1 + [F(q^*)/f(q^*)]'} = \frac{v''(q^*)}{v'(q^*)} - \frac{\mathrm{d}}{\mathrm{d}q}\log\left(1 + \frac{\mathrm{d}}{\mathrm{d}q}\left(\frac{F(q^*)}{f(q^*)}\right)\right);$$

consequently, $g''(q^*) \leq 0$ is equivalent to (18). □

If we assume that both v and f are twice continuously differentiable, it is easy to see that Lemma 3 implies Corollary 2.

[39] g is twice differentiable because we assume both v and f are.