Optimal Procurement Design: A Reduced Form Approach

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Abstract

Standard procurement models assume that the buyer knows the quality of the good at the time of procurement; however, in many settings, the quality is learned only long after the transaction. We study procurement problems in which the buyer's valuation of the supplied good depends directly on its quality, which is unverifiable and unobservable to the buyer. For a broad class of procurement problems, we identify procurement mechanisms maximizing any weighted average of the buyer's expected payoff and social surplus. The optimal mechanism can be implemented by an auction that restricts sellers to submit bids within specific intervals.

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1 Introduction

Procurement is essential for many different types of organizations. For example, hospitals rely on it to acquire medicines and medical devices (Bonaccorsi, Lyon, Pammolli, and Turchetti, 2000), and governments use it to obtain weapons and public services (Lalive, Schmutzler, and Zulehner, 2015). More generally, firms obtain most of their production inputs as well as many components of their product lines from procurement.

In many procurement contexts, the buyer's valuation of the supplied good depends directly on its quality. However, at the time the procurement contract is signed, the quality is often (at least partially) unverifiable in court and unobservable by the buyer. For example, this issue arises in both medical and government procurement, when acquiring medicines, medical devices, new weapons, or transportation systems. Although the buyer may have an estimate of the good's quality before signing the contract, the true quality often remains uncertain until long after the contract is executed. As pointed out by Manelli and Vincent (1995) and Lopomo, Persico, and Villa (2023), in such contexts, buyers who run standard procurement auctions may harbor "quality concerns"—that is, aggressively bidding sellers could be offering low-quality goods, potentially leading to undesirable outcomes.

In this paper, we ask the following question: how to optimally design the procurement mechanism when quality concerns are present? Under mild regularity conditions, a *bid-restricted auction* (BRA) turns out to be optimal for the buyer. A BRA is similar to a second-price auction, which induces an equilibrium in weakly dominant strategies. The primary difference is that sellers are restricted to bidding within a collection of intervals; see Figure 1 for an illustration. The seller with the lowest bid wins the auction and supplies the good, with ties broken uniformly at random. In most cases, the winning seller receives a payment equal to the second lowest bid; however, in special circumstances, a "payment reduction rule" is applied, resulting in a payment lower than the second lowest bid. This special rule helps eliminate sellers' incentives to take advantage of the "gaps" between the bid intervals.

More generally, we consider the problem of maximizing a weighted average of the buyer's expected payoff and the social surplus, subject to the requirement that the buyer's expected payoff is nonnegative. This constraint is particularly relevant in government procurement design, as the government typically aims to avoid running a deficit. In this more general setting, an optimal procurement mechanism is still a BRA, although a stochastic reserve price might need to be introduced.

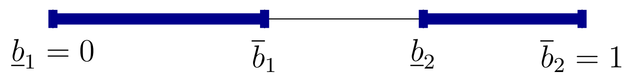

Figure 1. The illustration of a BRA. Sellers who wish to participate in the auction must submit a bid b within the two bid intervals, $[\underline{b}_1, \bar{b}_1]$ and $[\underline{b}_2, \bar{b}_2]$. In other words, bids strictly between $\bar{b}_1$ and $\underline{b}_2$ are not allowed.

The key insight is that the optimal BRA balances the tension between inducing price competition and alleviating quality concerns. Leveraging the structure of a second-price auction, a BRA fosters price competition within bid intervals, where quality concerns are mild. By prohibiting bids in the "gaps" between bid intervals—where quality concerns are severe—harmful price competition is eliminated. This reduces the likelihood of selecting a seller offering a low-quality good, thereby mitigating quality concerns. Put differently, while "global" quality concerns might necessitate abandoning competitive bidding altogether, "local" quality concerns can effectively be addressed by excluding certain bids from the auction.

We establish the optimality of BRA using a "reduced-form approach." Specifically, we transform the procurement design problem into choosing an interim allocation rule, also known as a "reduced form" in the literature, that maximizes a linear functional identified by some virtual surplus. An interim allocation rule specifies the expected probability that the buyer procures from each seller based on their reports of their private information. The interim allocation rule must satisfy a monotonicity constraint to ensure truthtelling and Border's (1991) condition to guarantee feasibility. We then apply techniques from linear optimization under a majorization constraint, developed by Kleiner, Moldovanu, and Strack (2021), to solve for the optimal interim allocation. We complete the argument by verifying that a BRA induces the optimal interim allocation.

The study of procurement problems with unverifiable and unobservable qualities is pioneered by Manelli and Vincent (1995). They point out that in many cases, a standard procurement auction may perform poorly due to quality concerns; instead, it may be optimal to sequentially render take-it-or-leave-it offers to potential sellers. Manelli and Vincent (2004) show that some "hybrid mechanisms," which combine elements of sequential offers and auctions, can be optimal in specific procurement settings. In contrast to these studies, our work identifies procurement mechanisms that maximize any weighted average of the buyer's expected payoff and social surplus across a broad class of procurement problems.

Lopomo, Persico, and Villa (2023) show that when the buyer's virtual surplus is single-peaked, a mechanism called the Lowball Lottery Auction (LoLA) is optimal. LoLA differs from a standard second-price procurement auction with a reserve price only in that sellers are not allowed to bid below a certain "floor price." Our work generalizes Lopomo et al. (2023) by identifying the optimal procurement mechanisms in a broader class of environments;[1] indeed, when the buyer's virtual surplus is single-peaked, the optimal BRA is a LoLA. More importantly, our results elucidate that what matters is not merely whether to use a floor price, but rather to disallow sellers from submitting bids in certain intervals where quality concerns are severe. Furthermore, Lopomo et al. (2023) do not impose the restriction that the buyer's expected payoff must be nonnegative, and therefore do not offer insights into how this constraint should be addressed, an issue that can be relevant in practice.[2]

The remainder of this paper is organized as follows. Section 2 sets up and transforms the optimal procurement design problem. For clearer insights, in Section 3 we study the special case of maximizing the buyer's expected payoff; Section 4 tackles the general problem of maximizing any weighted average of the buyer's payoff and the social surplus, with the constraint that the buyer's payoff has to be bounded below by zero. Section 5 discusses an extension of the main model as well as a dynamic implementation of the optimal mechanism. Section 6 concludes.

2 The procurement problem

The model, which is essentially the same as in Manelli and Vincent (1995, 2004), consists of one buyer and $n > 1$ symmetric potential sellers. The buyer would like to procure one unit of a product from one of the potential sellers. Each seller s has private information $q_s \in [0, 1]$. We will refer to q_s as the quality of the product offered by seller s; it can also be interpreted as seller s's cost or reservation value.[3] Qualities are independently and identically distributed according to a continuous density function f; we denote the corresponding cumulative distribution function by F. We also assume that $f(q) > 0$ for $q \in (0, 1]$, and $f(0) = 0$ only if $\lim_{q\to 0} (F(q)/f(q)) = 0$. Since we assume that the potential sellers are symmetric, we often suppress the subscript of a

[1]This is not merely a technical curiosity; violations of their assumption may naturally arise due to the nature of the procurement setting (see Example 1 for a concrete example).

[2]Interested readers are directed to Che (2008) for a survey of the procurement literature and Lopomo et al. (2023) for a review of recent developments on procurement design under quality concerns.

[3]These interpretations are further elaborated in Section 5.1.

seller's quality.

All agents in our model are expected utility maximizers. If the buyer procures the good from a seller, and a transfer t is made, the seller's payoff is $t - q$. If a seller does not sell, her payoff is zero. The buyer's valuation for a good of quality q is a continuous function $v(q)$; we assume that $v(0) \geq 0$. If the buyer makes a transfer t and receives an object of quality q, her payoff is $v(q) - t$; if no trade occurs, the buyer's payoff is zero.

By the revelation principle, it suffices to focus on direct mechanisms. A direct mechanism is characterized by a pair of functions $p_s : [0,1]^n \to [0,1]$ and $t_s : [0,1]^n \to \mathbb{R}$ for each seller s. If the sellers report $\boldsymbol{q} := (q_1, \dots, q_n)$, the buyer procures from seller s with probability $p_s(\boldsymbol{q})$, and she makes transfer $t_s(\boldsymbol{q})$ to seller s. Because the buyer wishes to acquire (at most) one unit of the product, for each $\boldsymbol{q} \in [0,1]^n$, the feasibility constraint must hold:

$$\sum_{s=1}^{n} p_s(\boldsymbol{q}) \leq 1; \tag{F}$$

(F) requires that the probability that the buyer buys from one of the potential sellers is less than or equal to 1.

If seller s reports q'_s and assumes that the rest of the sellers report truthfully, she would expect that the buyer procures from her with probability

$$P_s(q'_s) := \int p_s\left(q'_s, \boldsymbol{q}_{-s}\right) f^{n-1}\left(\boldsymbol{q}_{-s}\right) \mathrm{d}\boldsymbol{q}_{-s},$$

where $\boldsymbol{q}_{-s} := (q_1, \dots, q_{s-1}, q_{s+1}, \dots, q_n)$, and $f^{n-1}(\boldsymbol{q}_{-s}) := \prod_{k \neq s} f(q_k)$; she would expect to receive a monetary transfer of

$$T_s(q'_s) := \int t_s\left(q'_s, \boldsymbol{q}_{-s}\right) f^{n-1}\left(\boldsymbol{q}_{-s}\right) \mathrm{d}\boldsymbol{q}_{-s}.$$

We call $P_s(\cdot)$ the **interim allocation probability** for seller s. Then the expected payoff of seller s with quality q_s from reporting q'_s can be written as

$$\pi_s(q'_s \mid q_s) := T_s(q'_s) - q_s P_s(q'_s);$$

and we let $\pi_s(q_s) := \pi_s(q_s \mid q_s)$. We say that a direct mechanism $\{p_s, t_s\}_{s=1}^{n}$ is **incentive compatible** if for each seller s, all $q'_s \in [0,1]$, and (almost) all $q_s \in [0,1]$, $\pi_s(q_s) \geq \pi_s(q'_s \mid q_s)$; say that it is **individually rational for the sellers** if $\pi_s(q_s) \geq 0$ for each seller s and $q_s \in [0,1]$. Finally, the buyer's expected payoff under direct mechanism $\{p_s, t_s\}_{s=1}^n$ is

$$\pi_b := \sum_{s=1}^{n} \int_{[0,1]^n} \left[v\left(q_s\right) p_s(\boldsymbol{q}) - t_s(\boldsymbol{q})\right] f^n(\boldsymbol{q}) \, \mathrm{d}\boldsymbol{q}, \tag{1}$$

where $f^n(\boldsymbol{q}) := \prod_{s=1}^n f(q_s)$.

Lemma 1 characterizes the set of incentive compatible direct mechanisms and also eliminates transfers from the buyer's expected payoff. The proof is standard and hence omitted.

Lemma 1. *Let $\{p_s(\cdot)\}_{s=1}^n$ be a collection of allocation functions, where $p_s : [0,1]^n \to [0,1]$ satisfying (F).*

(1) There exists a collection of transfers $\{t_s(\cdot)\}_{s=1}^n$ such that $\{(p_s(\cdot), t_s(\cdot)\}_{s=1}^n$ is incentive compatible if and only if for each $s = 1, \ldots, n$, $P_s(\cdot)$ is decreasing.

(2) For any incentive compatible direct mechanism $\{(p_s(\cdot), t_s(\cdot)\}_{s=1}^n$,

i. it is individually rational for the sellers if and only if $\pi_s(1) \geq 0$; and

ii. the buyer's expected payoff is given by

$$\pi_b = \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s - \frac{F\left(q_s\right)}{f\left(q_s\right)}\right] p_s(\boldsymbol{q}) f^n(\boldsymbol{q}) \, \mathrm{d}\boldsymbol{q} - \sum_{s=1}^{n} \pi_s(1).$$

Without loss of generality, we can set $\pi_s(1) = 0$ for all $s = 1, \ldots, n$. Then by Lemma 1, the weighted average of the buyer's expected payoff (with weight γ) and the social surplus (with weight $1-\gamma$) can be written as

$$\begin{aligned} &\gamma \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s - \frac{F\left(q_s\right)}{f\left(q_s\right)}\right] p_s(\boldsymbol{q}) f^n(\boldsymbol{q}) \, \mathrm{d}\boldsymbol{q} + (1-\gamma) \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s\right] p_s(\boldsymbol{q}) f^n(\boldsymbol{q}) \, \mathrm{d}\boldsymbol{q} \\ &= \sum_{s=1}^{n} \int_0^1 \left[v\left(q_s\right) - q_s - \gamma \frac{F(q_s)}{f(q_s)}\right] p_s(\boldsymbol{q}) f^n(\boldsymbol{q}) \, \mathrm{d}\boldsymbol{q} \end{aligned} \tag{2}$$

Therefore, we consider the following maximization problem:

$$\begin{aligned}
\max_{\{p_s\}_{s=1}^n} \quad & \sum_{s=1}^n \int_0^1 \left[v\left(q_s\right) - q_s - \gamma \frac{F(q_s)}{f(q_s)} \right] p_s(\boldsymbol{q}) f^n(\boldsymbol{q}) \, \mathrm{d}\boldsymbol{q} \\
\text{subject to} \quad & \text{(F)} \\
& P_s(\cdot) \text{ is decreasing for each } s = 1, \ldots, n \\
& \sum_{s=1}^n \int_0^1 \left[v\left(q_s\right) - q_s - \frac{F\left(q_s\right)}{f\left(q_s\right)} \right] p_s(\boldsymbol{q}) f^n(\boldsymbol{q}) \, \mathrm{d}\boldsymbol{q} \geq 0.
\end{aligned}$$

The inequality requires that the buyer's expected payoff must be nonnegative.

When $\gamma = 1$, the above problem becomes the buyer's expected payoff maximization problem, which can be written as

$$\begin{aligned}
\max_{\{p_s\}_{s=1}^n} \quad & \sum_{s=1}^n \int_0^1 \left[v\left(q_s\right) - q_s - \frac{F\left(q_s\right)}{f\left(q_s\right)} \right] p_s(\boldsymbol{q}) f^n(\boldsymbol{q}) \, \mathrm{d}\boldsymbol{q} \\
\text{subject to} \quad & \text{(F)} \\
& P_s(\cdot) \text{ is decreasing for each } s = 1, \ldots, n.
\end{aligned}$$

The inequality constraint is not needed because setting $p_s(\boldsymbol{q}) = 0$ for all $\boldsymbol{q} \in [0,1]^n$ and for $s = 1, \ldots, n$ is feasible, ensuring that the value of the problem is nonnegative.

2.1 Detour: Majorization

Let $f, g \in L^1(0,1)$ be decreasing. Say that f **majorizes** g, denoted by $g \prec f$, if the following two conditions hold:

$$\int_0^x g(s) \, \mathrm{d}s \leq \int_0^x f(s) \, \mathrm{d}s \quad \text{for all } x \in [0,1], \tag{3}$$

$$\int_0^1 g(s) \, \mathrm{d}s = \int_0^1 f(s) \, \mathrm{d}s. \tag{4}$$

Say that f **weakly majorizes** g, denoted by $g \prec_w f$, if (3) holds (but not necessarily (4)). In what follows, we make use of some results on maximizing a linear functional under a majorization constraint developed by Kleiner et al. (2021). For readers' convenience, we include these results in Appendix A.

2.2 Transforming the problem: A reduced form approach

One novelty of this paper is that, instead of using a duality approach to characterize the optimal ex post allocation rule, we adopt a reduced form approach: we solve for the optimal interim allocation and identify a trading mechanism that implements it.

To this end, note that the objective function (2) can be written as, in terms of interim allocation probabilities,

$$\sum_{s=1}^{n} \int_{0}^{1} \left[v\left(q_s\right) - q_s - \gamma \frac{F\left(q_s\right)}{f\left(q_s\right)} \right] P_s(q_s) f(q_s) \, \mathrm{d}q_s.$$

Say that the collection of interim allocation probabilities $\{P_s\}_{s=1}^{n}$, where $P_s : [0,1] \to [0,1]$ for each s, is **implementable** if there exists a collection of allocation probabilities $\{p_s\}_{s=1}^{n}$ satisfying (F) that induces $\{P_s\}_{s=1}^{n}$ as its interim allocations; that is, for each $s = 1, \ldots, n$ and all $q_s \in [0,1]$,

$$P_s(q_s) = \int p_s\left(q_s, \boldsymbol{q}_{-s}\right) f^{n-1}\left(\boldsymbol{q}_{-s}\right) \, \mathrm{d}\boldsymbol{q}_{-s}.$$

Since the sellers are symmetric, it is without loss to restrict attention to symmetric interim allocations; thus, we can drop the subscript s from P_s and q_s, and write P and q instead. Consequently, the objective further reduces to

$$n \int_{0}^{1} \left[v(q) - q - \gamma \frac{F(q)}{f(q)} \right] P(q) f(q) \, \mathrm{d}q, \tag{5}$$

and the buyer's expected payoff can be written as

$$n \int_{0}^{1} \left[v(q) - q - \frac{F(q)}{f(q)} \right] P(q) f(q) \, \mathrm{d}q.$$

Consider the quantile $s = F(q)$, we define

$$\widetilde{P}(s) := P(F^{-1}(s))$$

as the **quantile interim allocation**. Let $\widetilde{P}^{*}(s) := (1-s)^{n-1}$; it is not difficult to see that $\widetilde{P}^{*}(\cdot)$ is the quantile interim allocation of the allocation rule that always procures from the seller with the lowest quality.

Border's (1991) celebrated theorem characterizes the set of implementable interim allocations. Lemma 2 translates Border's condition into majorization terminology.[4]

Lemma 2 (Border's condition)**.** *A decreasing interim allocation rule P is implementable if and only if the associated quantile interim allocation $\widetilde{P}(s)$ is weakly majorized by $\widetilde{P}^*$.*[5]

To simplify notation, let

$$h_\gamma(q) := v(q) - q - \gamma \frac{F(q)}{f(q)}$$

denote the integrand of (5) when the weight on the buyer's expected payoff is γ; we call h_γ the **weighted virtual surplus**. In particular, let

$$g(q) := h_1(q) = v(q) - q - \frac{F(q)}{f(q)}$$

denote the buyer's virtual surplus. By Lemma 2, the quantile interim allocation of a direct mechanism that maximizes the weighted average can be found by solving

$$\begin{aligned} \max_{\widetilde{P} \in \Omega_w(\widetilde{P}^*)} \quad & \int_0^1 h_\gamma(F^{-1}(s)) \widetilde{P}(s) \, \mathrm{d}s \\ \text{s.t.} \quad & \int_0^1 g(F^{-1}(s)) \widetilde{P}(s) \, \mathrm{d}s \geq 0, \end{aligned} \tag{6}$$

where

$$\Omega_w(\widetilde{P}^*) := \left\{ \widetilde{P} \in L^1(0,1) : \widetilde{P} \text{ is decreasing and } \widetilde{P} \prec_w \widetilde{P}^* \right\}.$$

A solution to problem (6) exists: $\Omega_w(\widetilde{P}^*)$ is compact by the Helly's selection theorem,[6] and therefore the constraint set

$$\Omega_w(\widetilde{P}^*) \cap \left\{ \widetilde{P} \in L^1(0,1) : \int_0^1 g(F^{-1}(s)) \widetilde{P}(s) \, \mathrm{d}s \geq 0 \right\}$$

is the intersection of a closed set and a compact set and hence also compact. The

[4] To the best of our knowledge, Gershkov, Goeree, Kushnir, Moldovanu, and Shi (2013) is the first paper that connects Border's condition to majorization (see their Footnote 4). We omit the proof of Lemma 2 since it can be proved by slightly modifying the proof of, for example, Theorem 1 in Hart and Reny (2015) or Theorem 3 in Kleiner et al. (2021).

[5] We use "increasing" and "decreasing" in the weak sense: "strict" will be added whenever needed.

[6] By "compact" we mean compact in the L^1 norm topology. For a complete proof of this fact, see the proof of Proposition 1 in Kleiner et al. (2021).

argument is completed by noting that the objective function is continuous.

In particular, the buyer's optimal interim allocation should solve the following problem:

$$\max_{\widetilde{P} \in \Omega_w(\widetilde{P}^*)} \int_0^1 g(F^{-1}(s))\widetilde{P}(s)\,\mathrm{d}s. \tag{7}$$

3 Buyer's optimal procurement mechanisms

For cleaner intuition and simpler notation, in this section we study a special case of our problem: the buyer's expected payoff maximization problem, where the welfare weight on the buyer is $\gamma = 1$. In Section 3.1, we solve the problem (7), and in Section 3.2 we identify a trading mechanism that implements the solution we find. All proofs in this section are relegated to Section B.

3.1 Buyer's optimal interim allocation

To ensure that the monotonicity constraint in problem (7) holds, the ironing technique (Myerson, 1981; Toikka, 2011) may be required. Define $\widetilde{g}(s) := g(F^{-1}(s))$ as the **quantile virtual surplus**, and let

$$G(s) := \int_0^s \widetilde{g}(x)\mathrm{d}x.$$

Let $\overline{G}$ be the concave hull of G: $\overline{G}(x) := \sup\{y : (x, y) \in \mathrm{co}(G)\}$, where $\mathrm{co}(G)$ is the convex hull of the graph of G. Equivalently, $\overline{G}$ is the pointwise smallest upper semicontinuous and concave function that lies above G (see, for example, Kamenica and Gentzkow, 2011). Call $\overline{g} := \overline{G}'$ the **ironed quantile virtual surplus**.[7] Adopting the convention that $\sup \varnothing = 0$, define

$$\overline{S} := \sup\{s \in [0, 1] : \overline{g}(s) \geq 0\}; \tag{8}$$

our assumptions on v and F guarantee that $\overline{S}$ is well-defined.

Proposition 1. *Let $\{[\underline{s}_i, \bar{s}_i)\}_{i \in \mathcal{I}}$ denote a collection of disjoint intervals with $[\underline{s}_i, \bar{s}_i) \subseteq [0, \overline{S}]$ for each $i \in \mathcal{I}$, such that*

[7] Because $\overline{G}$ is concave, it is differentiable almost everywhere. At points where it is not differentiable, we define $\overline{g}$ as the right derivative.

- $\overline{G}$ *is affine on* $[\underline{s}_i, \overline{s}_i)$ *for each* $i \in \mathcal{I}$, *and*
- $\overline{G} = G$ *on* $[0, \overline{S}] / \bigcup_{i \in \mathcal{I}} [\underline{s}_i, \overline{s}_i)$.

Then an optimal interim allocation $\hat{P}$ *satisfies*

$$\hat{P}(s) = \begin{cases} (1-s)^{n-1} & \text{if } s \in [0, \overline{S}] / \bigcup_{i \in \mathcal{I}} [\underline{s}_i, \overline{s}_i), \\ \frac{\int_{\underline{s}_i}^{\overline{s}_i} (1-s)^{n-1} \, \mathrm{d}s}{\overline{s}_i - \underline{s}_i} & \text{if } s \in [\underline{s}_i, \overline{s}_i), \\ 0 & \text{if } s \in (\overline{S}, 1]. \end{cases} \tag{9}$$

To understand Proposition 1, consider the auxiliary problem

$$\max_{\widetilde{P} \in \Omega_w(\widetilde{P}^*)} \int_0^1 \overline{g}(s) \widetilde{P}(s) \, \mathrm{d}s, \tag{10}$$

where the quantile virtual surplus $\widetilde{g}(s)$ in the original problem (7) is replaced by the ironed quantile virtual surplus $\overline{g}(s)$. For $s > \overline{S}$, the ironed virtual surplus is strictly negative, and thus the buyer should not trade. Consequently, any solution $\breve{P}$ of problem (10) must satisfy $\breve{P}(s) = 0$. For $s \leq \overline{S}$; however, to maximize the objective, we would like to set $\widetilde{P}$ higher at points where $\overline{g}$ is larger. Since $\overline{g}$ is decreasing, this means that we want to set $\widetilde{P}(s)$ higher when s is smaller. We see from (3) that no $\widetilde{P} \in \Omega_w(\widetilde{P}^*)$ attains a larger value than $\widetilde{P}^* = (1-s)^{n-1}$ itself for small s; thus, $\widetilde{P}^*$ solves the auxiliary problem (10). Moreover, on each $[\underline{s}_i, \overline{s}_i)$, since $\overline{G}$ is affine, $\overline{g}$ is constant, only the mean of the interim allocation on that interval matters. Therefore, letting

$$\hat{P}(s) = \mathbb{E}\left[\widetilde{P}^*(t) \;\middle|\; t \in [\underline{s}_i, \overline{s}_i)\right] = \frac{\int_{\underline{s}_i}^{\overline{s}_i} (1-s)^{n-1} \, \mathrm{d}s}{\overline{s}_i - \underline{s}_i}$$

on $[\underline{s}_i, \overline{s}_i)$ for each $i \in \mathcal{I}$ and letting $\hat{P} = \widetilde{P}^*$ otherwise would make $\hat{P}$ a solution of (10). In fact, if $\hat{P}$ solves the auxiliary problem (10), it also solves the original problem if and only if it is constant on $[\underline{s}_i, \overline{s}_i)$ for each $i \in \mathcal{I}$.[8] Therefore, interim allocation $\hat{P}$ defined by (9) solves problem (7), and thus is optimal.

Observe that ironing is required whenever the buyer's virtual surplus function has an increasing region; that is, there exists an interval $(x, y) \subseteq [0, 1]$ such that $g(q)$ is increasing on (x, y). While it might be standard to assume that F/f is increasing

[8] This is the pooling property of Myerson (1981) and Toikka (2011), which states that for all open intervals $I \subseteq [0, 1]$, $H(s) < \overline{H}(s)$ for all $s \in I$ implies that $\hat{P}$ must be constant on I.

(and hence $-F/f$ is decreasing), requiring that $g(q)$ is decreasing on $[0,1]$ also imposes strong restrictions on $v(q)$: roughly speaking, the buyer's marginal valuation of quality cannot be too high anywhere on the unit interval,[9] which implies that the buyer's quality concerns cannot be too strong even locally. Therefore, in our problem, ironing might be necessary, due to the nature of the procurement setting.

3.2 Implementation: Bid-restricted auctions

To find the buyer-optimal procurement mechanism, we need to find some trading mechanisms that implement the optimal interim allocation rule $\hat{P}(s)$ identified in Proposition 1. To this end, we introduce the following class of mechanisms.

Definition 1. *A **bid-restricted auction (BRA)** is a sealed bid auction with $M \in \mathbb{N}$ **bid intervals** $\{[\underline{b}_i, \overline{b}_i]\}_{i=1}^M$, where $\underline{b}_1 \geq 0$, $\overline{b}_M \leq 1$, and for all $i = 1, \ldots, M$, $\underline{b}_i \leq \overline{b}_i$ and $\overline{b}_i < \underline{b}_{i+1}$, with the following rules:*

- *Any seller who wishes to participate must submit a bid b within one of the bid intervals; i.e., $b \in \cup_{i=1}^M [\underline{b}_i, \overline{b}_i]$.*
- *The seller whose bid is the lowest wins the auction; in the event of a tie, the winning seller is chosen uniformly at random.*
- *If the winning bid is the only bid in its bid interval $[\underline{b}_i, \overline{b}_i]$ for some $i = 1, \ldots, M$, and if the second-lowest bid equals $\underline{b}_j$ for some $j > i$ with k other sellers bidding $\underline{b}_j$,[10] then the winning seller receives a payment of $(\underline{b}_j + k\overline{b}_{j-1})/(k+1)$.*
- *Otherwise, the winning seller receives a payment equal to the second-lowest bid.*

Intuitively, a BRA is similar to a second-price auction, with two key exceptions: (i) sellers are allowed to bid only within certain bid intervals; and (ii) for any bid interval that is not the highest, there is a "payment reduction rule." Under this rule, if the winning bid is the only bid in such a bid interval, the winning seller receives a payment that is determined by the other sellers' bids, which may be lower than the second-lowest bid. The following result establishes an important property of BRAs; to simplify the statement, we adopt the convention that $\overline{b}_0 := 0$.

[9]If we further assume that both v and f are differentiable, $g(q)$ is decreasing if and only if $v'(q) \leq 1 + (F(q)/f(q))'$ for all $q \in [0,1]$. For example, when qualities are uniformly distributed on $[0,1]$, $g(q)$ is decreasing if and only if $v'(q) \leq 2$ for all $q \in [0,1]$.

[10]If only one seller submits a bid, we take the second-lowest bid to be $\overline{b}_M$.

Lemma 3. *In a BRA, it is a weakly dominant strategy for any seller to not bid if her quality exceeds $\overline{b}_M$, bid her quality when $q \in [\underline{b}_i, \overline{b}_i)$ for some $i = 1, \ldots, M$, and bid $\underline{b}_i$ if $q \in [\overline{b}_{i-1}, \underline{b}_i)$ for some $i = 1, \ldots, M$.*

The structure inherited from a second-price auction and the "payment reduction rule" together imply that bidding as described in Lemma 3 is a weakly dominant strategy. Like a standard second-price auction, BRA also has the property that a seller's bid affects the payment she receives only indirectly, through its influence on the identity of the winner. Moreover, when the payment received by the winner differs from the second-lowest bid, it helps eliminate the sellers' incentives to take advantage of the "gaps" between the bid intervals.

To see this, consider a seller with quality $q \in [\overline{b}_{i-1}, \underline{b}_i)$. Because bids in the "gaps" are not allowed, it is more likely for someone to bid the lower bound of a bid interval. Therefore, the chance that some other sellers are also bidding $\underline{b}_i$ may not be negligible. In this case, if she bids $\underline{b}_i$ and it turns out to be the lowest bid, her chance of winning is at most $1/2$, since ties are broken equiprobably. This may incentivize her to underbid—for example, bidding just below $\overline{b}_{i-1}$—to increase her chances of winning. This is where the payment reduction rule becomes relevant: the reduced payment must be chosen carefully to eliminate the incentive for underbidding.[11]

A function h is said to be **structured** if there exists a finite partition of $[0, 1]$ into intervals on which h is either increasing or decreasing. If the virtual surplus g is structured, then there exist $L \in \mathbb{N}$ and a collection of disjoint intervals $\{[\underline{s}_i, \overline{s}_i)\}_{i=1}^{L}$ such that $\overline{G}$ is affine on each interval $[\underline{s}_i, \overline{s}_i)$ for $i = 1, \ldots, L$, and coincides with G on the complement $[0, \overline{S}] / \bigcup_{i=1}^{L} [\underline{s}_i, \overline{s}_i)$. We refer to the intervals $\{[\underline{s}_i, \overline{s}_i)\}_{i=1}^{L}$ as *pooling intervals*, and to the intervals comprising $[0, \overline{S}] / \bigcup_{i=1}^{L} [\underline{s}_i, \overline{s}_i)$ as *non-pooling intervals*; it is easy to see that there are no more than $L + 1$ non-pooling intervals.

Theorem 1. *If the virtual surplus g is structured, and $\overline{S} > 0$, a buyer-optimal procurement mechanism is a BRA whose bid intervals are determined by*

- *if $\underline{s}_1 = 0$, then $M = L$, $\underline{b}_i = F^{-1}(\overline{s}_i)$ for all $i = 1, \ldots, M$, $\overline{b}_i = F^{-1}(\underline{s}_{i+1})$ for $i = 1, \ldots, M - 1$, and $\overline{b}_M = F^{-1}(\overline{S})$;*
- *if $\underline{s}_1 > 0$, then $M = L + 1$, $\underline{b}_1 = 0$, $\underline{b}_i = F^{-1}(\overline{s}_{i-1})$ for $i = 2, \ldots, M$, $\overline{b}_i = F^{-1}(\underline{s}_i)$ for $i = 1, \ldots, M - 1$, and $\overline{b}_M = F^{-1}(\overline{S})$.*

[11] The reduced payment cannot be too low either, as doing so may encourage some sellers to overbid.

The optimality of BRA illustrates that prohibiting certain bids in an auction can alleviate quality concerns by eliminating vicious competition in prices, thereby "restoring" the auction's optimality. The two conditions in Theorem 1 guarantee that a bid-restricted auction maps every non-pooling interval to a bid interval, and every pooling interval to a gap between two bid intervals. Consequently, every seller with quality quantile in a non-pooling interval bids her quality, and every seller with quality quantile in a pooling interval bids the highest quality of that interval.

By Lemma 3, every seller with quality quantile s in a non-pooling interval wins if and only if the quality quantile of every other seller is above s, and therefore the interim allocation probability $\hat{P}(s) = (1-s)^{n-1}$. Furthermore, sellers with quality quantiles in the same pooling interval submit the same bid regardless of the exact quality quantiles, which indicates that the interim allocation probability is constant on the pooling intervals. Finally, if a seller has quality quantile above $\overline{S}$, she does not bid and hence never wins. In the proof, we formally show that the interim allocation induced by the BRA described in Theorem 1 is exactly the optimal interim allocation (9) in Proposition 1, which establishes the optimality of the described BRA.

In certain settings, the optimal BRA takes simple forms.

Corollary 1. *(i) (Manelli and Vincent, 1995)*

- *If the virtual surplus is decreasing, the BRA with one bid interval* $[0, F^{-1}(\overline{S})]$ *is optimal, which is equivalent to a standard second-price auction with reserve price* $F^{-1}(\overline{S})$*.*
- *If the virtual surplus is increasing, the BRA with a degenerate bid interval* $\{1\}$ *is optimal, which is equivalent to random allocation.*

(ii) (Lopomo et al., 2023) If the virtual surplus is single-peaked, the BRA with one bid interval $[\underline{b}, F^{-1}(\overline{S})]$ *is optimal.*

(iii) If the virtual surplus is single-dipped, the BRA with two bid intervals $[0, \overline{b}]$ *and* $\{1\}$ *is optimal.*

Corollary 1 (i) and (ii) indicate that the BRA nests the optimal procurement mechanisms identified in Manelli and Vincent (1995) and Lopomo et al. (2023) as special cases, respectively. Moreover, (iii) shows that the optimal mechanism when the virtual surplus is single-dipped is also simple.

Note that the payment reduction rule does not apply in Lopomo et al. (2023), as their assumptions limit the analysis to a single bid interval. In more general settings; however, the payment reduction rule is necessary to deter underbidding by sellers whose quality falls within a "gap" between two bid intervals. For example, the payment reduction rule remains relevant even when the virtual surplus is single-dipped.

4 Optimal procurement mechanisms

In this section, we show that under certain regularity conditions, a bid-restricted auction (with a stochastic reserve price if necessary) maximizes any weighted average of the buyer's expected payoff and the social surplus, subject to the constraint that the buyer's expected payoff remains nonnegative. It is not difficult to see that without this nonnegativity constraint, an analogue of Theorem 1 holds: as long as the weighted average h_γ is structured, a BRA is optimal. However, when the nonnegativity constraint is imposed, the problem becomes nontrivial. In Section 4.1 we solve problem (6) to get an optimal interim allocation, and we discuss how to implement it in Section 4.2. All proofs in this section are relegated to Appendix C.

4.1 Optimal interim allocation

To solve problem (6), we use a Lagrangian approach. Set up the Lagrangian with multiplier λ:

$$\mathcal{L}_\gamma = \int_0^1 \left[\widetilde{h}_\gamma(s) + \lambda \widetilde{g}(s)\right] \widetilde{P}(s)\,\mathrm{d}s,$$

where $\widetilde{h}_\gamma(s) := h_\gamma(F^{-1}(s))$ is the quantile version of the weighted virtual surplus. Define $\phi_\gamma(q;\lambda) := h_\gamma(q) + \lambda g(q)$, and let $\widetilde{\phi}_\gamma(s;\lambda) := \phi_\gamma(F^{-1}(s);\lambda)$. Evidently, $\widetilde{\phi}_\gamma(s;\lambda) = \widetilde{h}_\gamma(s) + \lambda\widetilde{g}(s)$, which is the quantile virtual surplus of the Lagrangian.

We solve the problem by maximizing the Lagrangian over all $\widetilde{P} \in \Omega_w(\widetilde{P}^*)$, and then find an appropriate Lagrangian multiplier such that the complimentary slackness condition holds. More specifically, we iron $\phi_\gamma(s;\lambda)$ to make sure that the monotonicity constraint holds: let

$$\Phi_\gamma(s;\lambda) := \int_0^s \widetilde{\phi}_\gamma(s;\lambda)\,\mathrm{d}x,$$

and

$$\overline{\phi}_\gamma(s;\lambda) := \frac{\partial}{\partial s}\,\overline{\Phi}_\gamma(s;\lambda)$$

is the ironed quantile virtual surplus of the Lagrangian. Similar to Proposition 1, for all s such that $\overline{\phi}_\gamma(s;\lambda) > 0$, the optimal interim allocation $\hat{P}$ is flat whenever ironing is needed, and coincide with $\widetilde{P}^*(s) = (1-s)^{n-1}$ otherwise. If there exists an interval $[S_+, S_0]$ on which $\overline{\phi}_\gamma(s;\lambda) = 0$, we may need to find some $\overline{P}$ satisfying

$$0 \leq \overline{P} \leq \frac{\int_{S_+}^{S_0}(1-s)^{n-1}\,\mathrm{d}s}{S_0 - S_+} := A(S_+, S_0)$$

and set $\hat{P}(s) = \overline{P}$ on $[S_+, S_0]$ to satisfy complementary slackness; the second inequality in the above expression is needed because Border's condition requires $\hat{P} \prec_w \widetilde{P}^*$.

Proposition 2. *If there exist $\lambda^* \geq 0$, $0 \leq S_+ \leq S_0 \leq 1$, and a collection of disjoint intervals $\{[\underline{s}_i, \bar{s}_i)\}_{i\in\mathcal{I}}$ with $[\underline{s}_i, \bar{s}_i) \subseteq [0, S_0]$ for each $i \in \mathcal{I}$, such that*

(i) $S_+ = \sup\left\{s \in [0,1] : \overline{\phi}_\gamma(s;\lambda^*) > 0\right\}$ *and* $S_0 = \sup\left\{s \in [0,1] : \overline{\phi}_\gamma(s;\lambda^*) \geq 0\right\}$*;*

(ii) $\overline{\Phi}_\gamma(s;\lambda^*)$ *is affine on* $[\underline{s}_i, \bar{s}_i)$ *for each* $i \in \mathcal{I}$ *and* $[S_+, S_0)$*, and*

(iii) $\overline{\Phi}_\gamma(s;\lambda^*) = \Phi_\gamma(s;\lambda^*)$ *on* $[0, S_+]/\bigcup_{i\in\mathcal{I}}[\underline{s}_i, \overline{s}_i)$*,*

then the interim allocation

$$\hat{P}(s) = \begin{cases} (1-s)^{n-1} & \text{if } s \in [0, S_+]/\bigcup_{i\in\mathcal{I}}[\underline{s}_i, \overline{s}_i) \\ \frac{\int_{\underline{s}_i}^{\overline{s}_i}(1-s)^{n-1}\mathrm{d}s}{\overline{s}_i - \underline{s}_i} & \text{if } s \in [\underline{s}_i, \overline{s}_i) \\ \overline{P} & \text{if } s \in (S_+, S_0] \\ 0 & \text{if } s \in (S_0, 1] \end{cases} \tag{11}$$

with $\overline{P} \in (0, A(S_+, S_0)]$ satisfying the complementary slackness condition

$$\lambda^* \int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s = 0$$

is optimal.

4.2 Implementation: Augmenting BRA if necessary

If the optimal interim allocation $\hat{P}$ defined in (11) satisfies $S_+ = S_0$, it takes exactly the same form as (9) in Proposition 1. Therefore, its implementation is described by Theorem 1, by replacing $\overline{S}$ by S_0. If, instead, $S_+ < S_0$, then to implement the optimal interim allocation we need to slightly modify the BRA. In this modified version, if a seller with quality $q \in (S_+, S_0]$ wins the auction, she supplies the good with probability $\overline{P}/A(S_+, S_0)$. The details of this modification are described below.

Definition 2. *An* ***augmented bid-restricted auction (aBRA)*** *is a sealed bid auction with*

- $M \in \mathbb{N}$ ***standard bid intervals*** $\{[\underline{b}_i, \bar{b}_i]\}_{i=1}^M$*, where* $\underline{b}_1 \geq 0$*,* $\bar{b}_M < 1$*, and for all* $i = 1, \ldots, M$*,* $\underline{b}_i \leq \bar{b}_i$ *and* $\bar{b}_i < \underline{b}_{i+1}$*,*
- *an* ***extra bid*** B *with* $\bar{b}_M < B \leq 1$*, and*
- *a* ***qualification rate*** $\zeta \in (0, 1]$*,*

with the following rules:

- *Any seller who wishes to participate must submit a bid* $b \in \cup_{i=1}^M [\underline{b}_i, \bar{b}_i] \cup \{B\}$*.*
- *With probability* $1 - \zeta$*, sellers who bid* B *are disqualified, meaning that their bids do not count.*
- *The seller whose bid is the lowest wins the auction; in the event of a tie, the winning seller is chosen uniformly at random.*
- *If the winning bid is the only bid in its bid interval* $[\underline{b}_i, \bar{b}_i]$ *for some* $i = 1, \ldots, M$*, and*
  - *if, furthermore, the second-lowest bid equals* $\underline{b}_j$ *for some* $j > i$ *with* k *other sellers bidding* $\underline{b}_j$*, then the winning seller receives a payment of* $(\underline{b}_j + k\bar{b}_{j-1})/(k + 1)$*;*
  - *if, furthermore, the second-lowest bid equals* B *with* k *other sellers bidding* B*,*[12] *then the winning seller receives a payment of* $\zeta(B + k\bar{b}_M)/(k + 1)$*.*
- *Otherwise, the winning seller receives a payment equal to the second-lowest bid.*

We modify a BRA in three ways to obtain an aBRA. First, we introduce an extra bid B that is strictly larger than the upper bound of the highest bid interval, $\bar{b}_M$. Second, any seller who submits the extra bid B is qualified to participate in the auction only with probability ζ, the qualification rate. Finally, if the second-lowest bid is B, the

[12]If only one seller submits a bid, we take the second-lowest bid to be B.

winning seller receives a reduced payment that is further adjusted by the qualification rate ζ, making sure that no seller with quality $q \in (\bar{b}_M, B]$ is incentivized to bid below $\bar{b}_M$ to take advantage of the guaranteed qualification.

An aBRA can be also interpreted as a BRA with $M+1$ bid intervals: $\{[\underline{b}_i, \bar{b}_i]\}_{i=1}^M$ and $\{B\}$, with a slightly adjusted payment rule and a *stochastic reserve price*. The stochastic reserve price takes the value B with probability ζ (the qualification rate) and the value $\bar{b}_M$ with complementary probability. This reserve price is determined only after the sellers have submitted their bids.

Theorem 2. *Let λ^* be the Lagrangian multiplier associated with the optimal interim allocation identified in Proposition 2. Suppose $\phi_\gamma(q;\lambda^*) = h_\gamma(q) + \lambda^* g(q)$ is structured, then*

(a) if $S_+ = S_0$, the BRA described in Theorem 1 with $\overline{S}$ replaced by S_0 is optimal.

(b) if $S_+ < S_0$, an aBRA with standard bid intervals as described in Theorem 1 with $\overline{S}$ replaced by S_+, extra bid $F^{-1}(S_0)$, and qualification rate $\zeta = \overline{P}/A(S_+, S_0)$ is optimal.

Corollary 2 and Corollary 3 identify environments in which some simple trading mechanisms are optimal.

Corollary 2. *Suppose $v(q) - q$ is strictly decreasing, F is twice continuously differentiable, and both F and $1-F$ are log-concave. Then a second-price auction with reserve price $F^{-1}(\overline{S})$ is optimal.*

Corollary 2 specifies that if the buyer values marginal quality uniformly less than the potential sellers, under standard distributional assumptions in the mechanism design literature, a second-price auction (with a reserve price if needed) maximizes any weighted average of the buyer's expected payoff and the social surplus.

Corollary 3. *Suppose $v(q)$ is concave, and F/f is convex in q, then $\phi_\gamma(q;\lambda^*)$ is single-peaked for all $\lambda^* \geq 0$. Consequently,*

- *if there do not exist $0 \leq S_+ < S_0 \leq 1$ such that $\phi_\gamma(\cdot;\lambda^*) = 0$ on $[S_+, S_0]$, the a BRA with one bid interval is optimal;*
- *otherwise, an aBRA with one standard bid interval $[\underline{b}, F^{-1}(S_+)]$ and extra bid $F^{-1}(S_0)$ is optimal.*

Assuming that v is concave implies that the buyer's marginal valuation of quality is decreasing. As pointed out by Lopomo et al. (2023), the convexity of F/f is satisfied by many familiar distributions with bounded supports, including power distributions,[13] (truncated) Pareto distributions, (truncated) exponential distributions. In fact, it is also satisfied by Beta distributions with both parameters greater than or equal to 1.

5 Discussion

5.1 An extension of the main model

In the main model, the buyer's valuation is assumed to be a deterministic function of the sellers' quality. Moreover, a seller's cost, or her reservation value, is identical to her quality. These assumptions are made to simplify notation and emphasize the quality concerns. In many relevant applications; however, it might be more natural to assume that the buyer's valuation is a random variable, and/or take the sellers' costs as primitives. In what follows, we show that, even in these settings, focusing on the main model is without loss.[14]

For concreteness, consider a buyer who would like to contract with one of several potential suppliers to develop a new project, say a new production line. The cost of supplier s, $c_s \in [0,1]$, is her private information; the costs are independently and identically distributed according to a continuous density function $f_C(\cdot)$. The project's value is not perfectly revealed to the buyer until the end of the development phase at the earliest, which is long after penning the contract. Consequently, at the time of contracting the buyer's valuation is a random variable Ξ. We assume that the realization of Ξ, $\xi \in [\underline{\xi}, \overline{\xi}]$, is not contractable. The buyer believes that Ξ and C are correlated, and that the conditional distribution of Ξ is $f_{\Xi|C}(\cdot|c)$. One possible assumption can be that Ξ and C are positively affiliated, or equivalently MTP_2 (Karlin and Rinott, 1980; see also Milgrom and Weber, 1982).

Let $\boldsymbol{c} = (c_1, \ldots, c_n)$. Given a direct mechanism $\{p_s(\boldsymbol{c}), t_s(\boldsymbol{c})\}_{s=1}^n$, where for each cost profile $\boldsymbol{c}$, $p_s(\boldsymbol{c})$ specifies the probability that the buyer contracts with supplier s, and $t_s(\boldsymbol{c})$ is the transfer that the buyer pays to supplier s, the buyer's expected payoff

[13] The CDF of a power distribution takes the form of $F(x) = x^\alpha$, where $\alpha > 0$. When $\alpha = 1$, we get the uniform distribution.

[14] For brevity, in Section 5 we only discuss the buyer's optimal problem; extending the analysis to allow for any weighted average is straightforward.

can be written as

$$\begin{aligned}\pi_b &= \int_{[0,1]^n} \left[\sum_{s=1}^{n} \int_{\underline{\xi}}^{\bar{\xi}} (\xi\, p_s(\boldsymbol{c}) - t_s(\boldsymbol{c})) f_{\Xi\,|C}(\xi|c_s)\, \mathrm{d}\xi \right] f^n(\boldsymbol{c})\, \mathrm{d}\boldsymbol{c} \\ &= \int_{[0,1]^n} \sum_{s=1}^{n} \left[\left(\int_{\underline{\xi}}^{\bar{\xi}} \xi f_{\Xi\,|C}(\xi|c_s)\, \mathrm{d}\xi \right) p_s(\boldsymbol{c}) - t_s(\boldsymbol{c}) \right] f^n(\boldsymbol{c})\, \mathrm{d}\boldsymbol{c} \\ &= \int_{[0,1]^n} \sum_{s=1}^{n} \left(\mathbb{E}[\Xi \,|\, C = c_s]\, p_s(\boldsymbol{c}) - t_s(\boldsymbol{c}) \right) f^n(\boldsymbol{c})\, \mathrm{d}\boldsymbol{c},\end{aligned}$$

where $f^n(\boldsymbol{c}) := \prod_{s=1}^{n} f_C(c_s)$. If we define $v(c_s) := \mathbb{E}[\Xi \,|\, C = c_s]$, we see from (1) that the problem here is identical to the procurement problem we study above, and the curvature of $v(c_s)$ is governed by the conditional distribution. For example, if Ξ and C are positively affiliated, $v(\cdot)$ is increasing.

Example 1. A manufacturer would like to procure a machine for production. For simplicity, suppose that her valuation is identical to the durability of the machine. She believes that the two potential sellers' costs are identically, independently, and uniformly distributed. Conditional on the cost realization c, her valuation Ξ is distributed according to a Pareto distribution with scale 0.5 and shape $2.2 - c$.[15] Consequently,

$$v(c) = \mathbb{E}[\Xi \,|\, C = c] = \frac{1.1 - 0.5c}{1.2 - c},$$

and $g(c) = v(c) - 2c$. We plot g and $\overline{g}$ in Figure 2a. By Proposition 1, the optimal interim allocation is given by

$$\hat{P}(c) = \begin{cases} 1 - c & c < 0.48, \\ 0.26 & c \geq 0.48, \end{cases}$$

which is shown in Figure 2b. By Corollary 1, a BRA with two bid intervals $[0, 0.48]$ and $\{1\}$ is optimal.

5.2 Dynamic Implementation

A BRA can be implemented with a descending clock auction format with irrevocable exit and "clock jumps." Specifically, the clock is lowered continuously starting from $\bar{b}_M$.

[15] The scale parameter can be interpreted as the length of the machine's warranty.

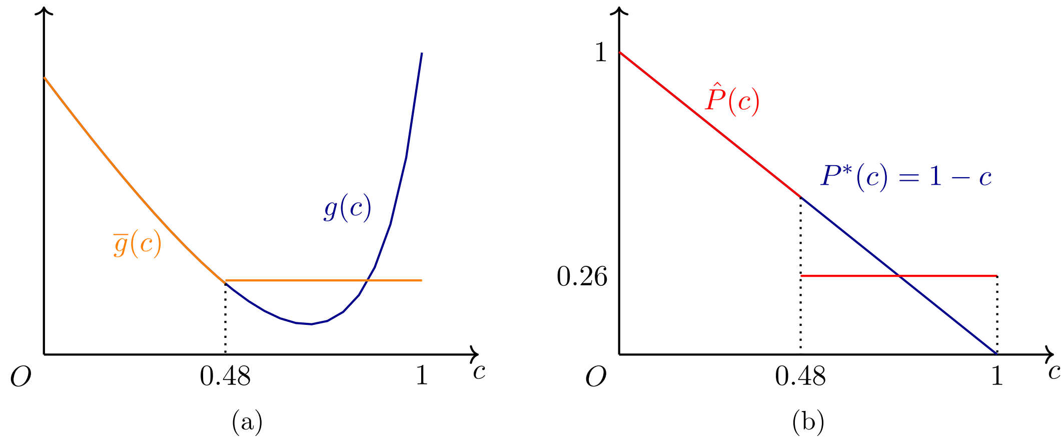

FIGURE 2. In panel (a), the blue curve is the buyer's quantile virtual surplus $\widetilde{g}$, and the orange curve is the ironed quantile virtual surplus $\overline{g}$. In panel (b), the blue curve is $P^*(c) = 1 - c$ that appears in Border's condition, and the orange curve is the optimal interim allocation $\hat{P}(c)$.

For each $i = 2, \dots, M$, if the clock reaches $\underline{b}_i$ and there are still sellers remaining, the auctioneer announces that the clock will jump to $\overline{b}_{i-1}$; sellers may choose to exit exactly when the clock jumps. The clock is lowered until one of the following occurs: only one seller remains; the remaining sellers all exit simultaneously; or the clock reaches $\underline{b}_1$, at which point all remaining sellers must exit. In the first case, the remaining seller wins the auction; in the last two cases, the winner is selected uniformly at random from the last few sellers exited simultaneously.

If a seller wins the auction by being the last remaining participant after $k \geq 1$ sellers exit when the clock jumps at $\underline{b}_i$ for some $i = 2, \dots, M$, she receives a payment of $(\underline{b}_i + k\overline{b}_{i-1})/(k+1)$; if the final exit did not occur at a clock jump, she receives a payment equal to the clock value at which the clock stopped. If a seller wins the auction by being one of the last sellers who exited simultaneously, she receives a payment equal to the clock value at the time of exit.

6 Conclusion

We explored procurement design problems in which the buyer's valuation of the good supplied depends directly on its quality, which is both unverifiable and unobservable. We analyzed the problem of maximizing an arbitrary weighted average of the buyer's expected payoff and the social surplus, subject to the constraint that the buyer's expected payoff remains nonnegative. To tackle this problem, we employed a novel reduced-form approach utilizing techniques from linear optimization under a majorization constraint. We found that a *bid-restricted auction*—a mechanism similar to a second-price auction, featuring a dominant strategy equilibrium but restricting sellers to bidding within specified intervals—is optimal.

In our analysis, we abstract from collusion, repeated interaction, and endogenous seller entry to isolate the impact of quality concerns in procurement. Exploring the optimal procurement mechanism under these concerns could be an interesting direction for future research.

Appendix A Results on majorization

Denote the set of decreasing functions in $L^1(0,1)$ that are majorized by f by

$$\Omega(f) := \{g \in L^1(0,1) : g \text{ is decreasing, } g \prec f\};$$

similarly, denote the "weak majorization set" by

$$\Omega_w(f) := \{g \in L^1(0,1) : g \text{ is decreasing, } g \prec_w f\}.$$

The following results are taken from Kleiner et al. (2021) and modified to our environment. Let A be an arbitrary subset of a topological vector space, we denote the set of its extreme points by $\mathrm{ext}A$.

Theorem 3 (Theorem 1 in Kleiner et al., 2021)**.** *Let $f \in L^1(0,1)$ be decreasing. Then $h \in \mathrm{ext}\Omega(f)$ if and only if there exists a collection of disjoint intervals $[\underline{x}_i, \overline{x}_i)$ indexed by $i \in I$ such that for almost all $x \in [0,1]$,*

$$h(x) = \begin{cases} f(x) & \text{if } x \notin \bigcup_{i\in I} [\underline{x}_i, \overline{x}_i) \\ \frac{\int_{\underline{x}_i}^{\overline{x}_i} f(s)\mathrm{d}s}{\overline{x}_i - \underline{x}_i} & \text{if } x \in [\underline{x}_i, \overline{x}_i)\,. \end{cases}$$

For $B \subseteq [0,1]$, denote by $\mathbf{1}_B(x)$ the indicator function of B: it equals 1 if $x \in B$ and 0 otherwise.

Corollary 4 (Corollary 2 in Kleiner et al., 2021)**.** *Let $f \in L^1(0,1)$ be decreasing. Then $h \in \mathrm{ext}\Omega_w(f)$ if and only if there exists $\theta \in [0,1]$ such that $h \in \mathrm{ext}\Omega(f \cdot \mathbf{1}_{[0,\theta]})$ and $h(x) = 0$ for almost all $x \in (\theta, 1]$.*

Now consider the problem

$$\max_{m\in\Omega(f)} \int_0^1 c(x)r(x)\,\mathrm{d}x, \tag{12}$$

where $f \in L^1(0,1)$ is strictly decreasing, and c is a bounded function. Define

$$C(x) = \int_0^x c(s)\,\mathrm{d}s,$$

and let $\overline{C}$ be its concave hull. Proposition 3 characterizes a solution to problem (12).

Proposition 3 (Proposition 2 in Kleiner et al., 2021)**.** *Let $h \in \mathrm{ext}\Omega(f)$, and let $\{[\underline{x}_i, \bar{x}_i) : i \in I\}$ be the collection of intervals described in Theorem 3. Then h is optimal if and only if $\overline{C}$ is affine on $[\underline{x}_i, \bar{x}_i)$ for each $i \in I$ and $\overline{C} = C$ otherwise.*

Appendix B Proofs for Section 3

B.1 Proof of Proposition 1

Because the objective function of problem (7) is linear, by Bauer's maximum principle (Aliprantis and Border (2006), Theorem 7.69, page 298), the maximum is attained at an extreme point $\hat{P}$ of $\Omega_w(\widetilde{P}^*)$. By Corollary 4, there exists $\bar{s} \in [0,1]$ such that $\hat{P}$ is an extreme point of $\Omega(\widetilde{P}^* \cdot \mathbf{1}_{[0,\bar{s}]})$ and equals zero on $[\bar{s}, 1]$. Furthermore, the optimality of $\hat{P}$ requires that the quality $\bar{q} = F^{-1}(\bar{s})$ must satisfy $\overline{g}(\bar{q}) = 0$; thus, setting $\bar{s} = \sup\{s \in [0,1] : \overline{g}(s) \geq 0\} = \overline{S}$ suffices. Then Theorem 3 implies that $\hat{P}$ must take the form of

$$\hat{P}(s) = \begin{cases} (1-s)^{n-1} & \text{if } s \in [0,\overline{S}] / \bigcup_{i\in\mathcal{I}} [\underline{s}_i, \overline{s}_i)\,, \\ \frac{\int_{\underline{s}_i}^{\overline{s}_i} (1-s)^{n-1}\mathrm{d}s}{\overline{s}_i - \underline{s}_i} & \text{if } s \in [\underline{s}_i, \overline{s}_i)\,, \\ 0 & \text{if } s \in (\overline{S}, 1]; \end{cases}$$

and by Proposition 3, the collection $\{[\underline{s}_i, \overline{s}_i) \subseteq [0,\overline{S}] : i \in \mathcal{I}\}$ is pinned down by the intervals on which $\overline{G}$ is affine.

B.2 Proof of Lemma 3

First, consider a seller with $q \geq \bar{b}_M$. Because the highest allowable bid is $\bar{b}_M$, if she ever wins, her payoff is at most zero, which is no better than not bidding regardless of what other sellers do.

Next, consider a seller has $q \in (\underline{b}_i, \bar{b}_i)$ for some $i = 1, \ldots, M$; let $b_{-}^{\min}$ denote the minimum bid among all other sellers. By bidding exactly q, there are the following cases:

- if $b_{-}^{\min} < q$, the seller's payoff is 0;
- if $b_{-}^{\min} \in \left(q, \bar{b}_i\right]$ or $b_{-}^{\min} \in \left(\underline{b}_j, \bar{b}_j\right]$ for some $j > i$, the seller's payoff is $b_{-}^{\min} - q$;
- if $b_{-}^{\min} = \underline{b}_j$ for some $j > i$, and there are k other sellers bid $b = b_{-}^{\min}$, then the seller's payoff is $(\underline{b}_j + k\bar{b}_{j-1})/(k+1) - q$.

If the seller bids $b < q$ instead, when $b \leq b_{-}^{\min} < q$ she gets a strictly negative payoff instead of 0, and otherwise she gets the same payoff as bidding q. If the seller bids $b > q$, when $b_{-}^{\min} < b$ she gets zero, and when $b_{-}^{\min} \geq b$ her payoff is otherwise identical to bidding $b = q$, except for the case that $b_{-}^{\min} = b = \underline{b}_j$ for some $j > i$. In this case, suppose k other sellers bid $\underline{b}_j$, then the seller's (expected) payoff is $(\underline{b}_j - q)/(k+1)$. But by bidding $b = q$, fixing other sellers' bids, her payoff is $(\underline{b}_j + k_j\bar{b}_{j-1})/(k+1) - q$, and

$$\frac{\underline{b}_j + k\bar{b}_{j-1}}{k+1} - q - \frac{\underline{b}_j - q}{k+1} = \frac{k\left(\bar{b}_{j-1} - q\right)}{k+1} \geq 0,$$

where the inequality holds because $i \leq j-1$. Thus, for a seller with $q \in (\underline{b}_i, \bar{b}_i)$ for some $i = 1, \ldots, M$, bidding q is a weakly dominant strategy.

Now consider a seller with $q \in [\bar{b}_{i-1}, \underline{b}_i]$ for some $i = 1, \ldots, M$ (recall that we set $\bar{b}_0 := 0$). By bidding $\underline{b}_i$, there are the following cases:

- if $b_{-}^{\min} \leq \bar{b}_{i-1}$, the seller's payoff is 0;
- if $b_{-}^{\min} = \underline{b}_i$, the seller's payoff is $(\underline{b}_i - q)/(k+1)$ if k other sellers bid b_i;
- if $b_{-}^{\min} \in \left(\underline{b}_j, \bar{b}_j\right]$ for same $j \geq i$, the seller's payoff is $b_{-}^{\min} - q$;
- if $b_{-}^{\min} = \underline{b}_j$ for some $j > i$, and there are k other sellers bid $b = b_{-}^{\min}$, then the seller's payoff is $(\underline{b}_j + k_j\bar{b}_{j-1})/(k+1) - q$.

If the seller bids $b \le \bar{b}_{i-1}$ instead, she loses when $\underline{b}^{\min} < b$ and get zero payoff; when $b \le \underline{b}^{\min} \le \bar{b}_{i-1}$, the seller payoff is bounded above by zero. When $\underline{b}^{\min} = \underline{b}_i$, and k other sellers bid $\underline{b}_i$, the seller gets $(\underline{b}_i + k\bar{b}_{i-1})/(k+1) - q$; but the payoff from bidding $\underline{b}_i$ is $(\underline{b}_i - q)/(k+1)$, and

$$\left(\frac{\underline{b}_i + k\bar{b}_{i-1}}{k+1} - q\right) - \frac{\underline{b}_i - q}{k+1} = \frac{k\left(\bar{b}_{i-1} - q\right)}{k+1} \le 0,$$

where the inequality holds because $q \ge \bar{b}_{i-1}$. Otherwise, the seller gets the same payoff as bidding $\underline{b}_i$. If the seller bids $b > \underline{b}_i$, her payoff is 0 if $\underline{b}^{\min} < b$, and if $\underline{b}^{\min} \ge b$, the seller gets the same payoff as bidding $\underline{b}_i$ unless $\underline{b}^{\min} = b = \underline{b}_j$ for some $j > i$. In this case, suppose k other sellers bid $\underline{b}_j$, the seller's payoff is $(\underline{b}_j - q)/(k+1)$; but in this case, by bidding $\underline{b}_i$ the seller gets $(\underline{b}_j + k\bar{b}_{j-1})/(k+1) - q$, which is no lower. Therefore, for a seller with $q \in [\bar{b}_{i-1}, \underline{b}_i]$ for some $i = 1, \ldots, M$, bidding $\underline{b}_i$ is a weak dominant strategy.

Finally, notice that the argument above also establishes that bidding the quality is a weakly dominant strategy if the seller's quality is $\underline{b}_i$ for some $i = 1, \ldots, M$. This completes the proof.

B.3 Proof of Theorem 1

The description of the BRA in the statement of the theorem and Lemma 3 together indicate that

- If a seller's quality quantile $s \in [0, \overline{S}) \big/ \bigcup_{i=1}^{L} [\underline{s}_i, \overline{s}_i)$ (that is, in a non-pooling interval), she bids her quality: $b = q = F^{-1}(s)$.
- If a seller's quality quantile $s \in [\underline{s}_i, \overline{s}_i)$ for some $i = 1, \ldots, L$ (that is, in a pooling interval), she bids the lower bound of a bid interval.
- If a seller's quality quantile $s \ge \overline{S}$ for some $i = 1, \ldots, L$, she does not bid.

Therefore, for $s \ge \overline{S}$, the interim allocation probability $P_{BRA}(s)$ is zero. Moreover, by the definition of a BRA, a seller with quality quantile $s \in [0, \overline{S}) \big/ \bigcup_{i=1}^{L} [\underline{s}_i, \overline{s}_i)$ wins if and only if all other sellers have quality quantiles above s, which happens with probability $(1-s)^{n-1}$. Consequently, $P_{BRA}(s) = (1-s)^{n-1}$ for all $s \in [0, \overline{S}) \big/ \bigcup_{i=1}^{L} [\underline{s}_i, \overline{s}_i)$.

Now consider a seller with quality quantile $s \in [\underline{s}_i, \overline{s}_i)$ for some $i = 1, \ldots, L$. She could only win the auction when there is no seller with $s < \underline{s}_i$; there are the following cases:

- All other $n-1$ sellers have $s \geq \overline{s}_i$. This case happens with probability $(1-\overline{s}_i)^{n-1}$, and in this case this seller wins with probability 1.
- One other seller has $s \in [\underline{s}_i, \overline{s}_i)$, and $n-2$ other sellers have $s \geq \overline{s}_i$. This case happens with probability $\binom{n-1}{1}(1-\overline{s}_i)^{n-2}(\overline{s}_i - \underline{s}_i)$, and in this case this seller wins with probability $1/2$.
- Two other seller has $s \in [\underline{s}_i, \overline{s}_i)$, and $n-3$ other sellers have $s \geq \overline{s}_i$. This case happens with probability $\binom{n-1}{2}(1-\overline{s}_i)^{n-3}(\overline{s}_i - \underline{s}_i)^2$, and in this case this seller wins with probability $1/3$.
- ...
- $n-2$ other sellers have $s \in [\underline{s}_i, \overline{s}_i)$, and one other seller has $s \geq \overline{s}_i$. This case happens with probability $\binom{n-1}{n-2}(1-\overline{s}_i)(\overline{s}_i - \underline{s}_i)^{n-2}$, and in this case this seller wins with probability $1/(n-1)$.
- All other sellers have $s \in [\underline{s}_i, \overline{s}_i)$. This case happens with probability $\binom{n-1}{n-1}(\overline{s}_i - \underline{s}_i)^{n-1}$, and in this case this seller wins with probability $1/n$.

Therefore, the interim allocation probability for this seller with quality quantile $s \in [\underline{s}_i, \overline{s}_i)$ is given by

$$(1-\overline{s}_i)^{n-1} + \binom{n-1}{1}\frac{(1-\overline{s}_i)^{n-2}(\overline{s}_i - \underline{s}_i)}{2} + \binom{n-1}{2}\frac{(1-\overline{s}_i)^{n-3}(\overline{s}_i - \underline{s}_i)^2}{3} + \\ \cdots + \binom{n-1}{n-2}\frac{(1-\overline{s}_i)(\overline{s}_i - \underline{s}_i)^{n-2}}{n-1} + \binom{n-1}{n-1}\frac{(\overline{s}_i - \underline{s}_i)^{n-1}}{n}. \tag{13}$$

Observing that for every $k = 0, \ldots, n-1$,

$$\frac{1}{k}\binom{n-1}{k} = \frac{1}{n}\binom{n}{k+1};$$

(13) can therefore be written as

$$\frac{1}{n}\binom{n}{0}(1-\overline{s}_i)^{n-1} + \frac{1}{n}\binom{n}{1}\frac{(1-\overline{s}_i)^{n-2}(\overline{s}_i - \underline{s}_i)}{2} + \frac{1}{n}\binom{n}{2}\frac{(1-\overline{s}_i)^{n-3}(\overline{s}_i - \underline{s}_i)^2}{3} + \\ \cdots + \frac{1}{n}\binom{n}{n-1}\frac{(1-\overline{s}_i)(\overline{s}_i - \underline{s}_i)^{n-2}}{n-1} + \frac{1}{n}\binom{n}{n}\frac{(\overline{s}_i - \underline{s}_i)^{n-1}}{n},$$

which is equivalent to

$$
\begin{aligned}
&\frac{\binom{n}{1}\left(1-\overline{s}_i\right)^{n-1}\left(\overline{s}_i-\underline{s}_i\right)+\binom{n}{2}\left(1-\overline{s}_i\right)^{n-2}\left(\overline{s}_i-\underline{s}_i\right)^2+\cdots+\binom{n}{n}\left(\overline{s}_i-\underline{s}_i\right)^n}{n\left(\overline{s}_i-\underline{s}_i\right)}\\
=&\frac{\binom{n}{0}\left(1-\overline{s}_i\right)^{n}+\binom{n}{1}\left(1-\overline{s}_i\right)^{n-1}\left(\overline{s}_i-\underline{s}_i\right)+\cdots+\binom{n}{n}\left(\overline{s}_i-s_i\right)^n-\binom{n}{0}\left(1-\overline{s}_i\right)^{n}}{n\left(\overline{s}_i-\underline{s}_i\right)}\\
=&\frac{\left(1-\underline{s}_i\right)^{n}-\left(1-\overline{s}_i\right)^{n}}{n\left(\overline{s}_i-\underline{s}_i\right)}=\frac{\int_{\underline{s}_i}^{\overline{s}_i}(1-s)^{n-1}\,\mathrm{d}s}{\overline{s}_i-\underline{s}_i},
\end{aligned}
$$

where the second equality follows from the binomial theorem, and the third equality follows from the fundamental theorem of calculus. Thus, $P_{BRA}(s)=\frac{\int_{\underline{s}_i}^{\overline{s}_i}(1-s)^{n-1}\,\mathrm{d}s}{\overline{s}_i-\underline{s}_i}$ on $[\underline{s}_i,\overline{s}_i)$ for every $i=1,\ldots,L$.

Summing up, we have established that the interim allocation probability induced by the BRA described in the statement of Theorem 1, P_{BRA}, equals to the interim allocation probability $\hat{P}$ defined by (9) in Proposition 1 F-almost everywhere, which implies that the BRA described in the statement of the theorem is indeed a buyer-optimal procurement mechanism.

Appendix C Proofs for Section 4

C.1 Proof of Proposition 2

Let

$$H_\gamma(s)=\int_0^s\widetilde{h}_\gamma(x)\,\mathrm{d}x\quad\text{and}\quad\Phi_\gamma(s;\lambda)=\int_0^s\widetilde{\phi}_\gamma(s;\lambda)\,\mathrm{d}x,$$

and let $\overline{H}_\gamma$ and $\overline{\Phi}_\gamma$ be the concave hulls of H_γ and Φ_γ, respectively. We further define $\{[\underline{x}_i,\overline{x}_i)\}_{i\in I}$ and $\{[\underline{y}_i,\overline{y}_i)\}_{i\in J}$ be the collections of intervals on which $\overline{H}_\gamma$ and $\overline{\Phi}_\gamma$ are affine, respectively. Now let

$$\overline{h}_\gamma(s)=\overline{H}'_\gamma(s),\quad\text{and}\quad\overline{\phi}_\gamma(s;\lambda)=\frac{\partial}{\partial s}\overline{\Phi}_\gamma(s;\lambda);$$

$\overline{h}_\gamma(s)$ and $\overline{\phi}_\gamma(s;\lambda)$ are the ironed quantile social surplus and the ironed quantile virtual surplus of the Lagrangian, respectively. By construction, both are decreasing in s. Define $Z:=\{s\in[0,1]:\overline{h}_\gamma(s)=0\}$; Z is the set of points on which the ironed quantile social surplus is zero. There are two cases in which Z is empty: either $\overline{h}_\gamma(1)>0$, or $\overline{h}_\gamma(0)<0$. Because $\overline{h}_\gamma(0)<0$ represents the uninteresting case that it is undesirable to trade under incomplete information, we assume that if Z is empty, we have $\overline{h}_\gamma(1) > 0$.

By Corollary 1 on page 219 and Theorem 2 on page 221 of Luenberger (1969), $\hat{P} \in \Omega_w(\widetilde{P}^*)$ solves problem (6) if and only if there exists $\lambda \geq 0$ such that $\hat{P}$ maximizes $\mathcal{L}$, and the complementary slackness condition

$$\lambda \int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s = 0$$

holds. Consequently, an optimal $\hat{P}$ can be found using the following algorithm:

Step 1. Check that if there exists $\hat{s} \in [0,1]$, either $\hat{s} \in Z$, or $Z = \varnothing$ and $\hat{s} = 1$ such that

$$\hat{P}(s) = \begin{cases} (1-s)^n & \text{if } s \in [0,\hat{s}]/\bigcup_{i\in I}\,[\underline{x}_i, \overline{x}_i) \\ \frac{\int_{\underline{x}_i}^{\overline{x}_i}(1-s)^{n-1}\mathrm{d}s}{\overline{x}_i - \underline{x}_i} & \text{if } s \in [\underline{x}_i, \overline{x}_i) \\ 0 & \text{if } s \in (\hat{s}, 1] \end{cases}$$

satisfies

$$\int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s \geq 0.$$

If so, we can set $\lambda = 0$, which implies that the quantile virtual surplus of the Lagrangian coincides with the ironed quantile social surplus: $\overline{\phi}_\gamma(s;0) = \overline{h}_\gamma(s)$. Setting $S_+ = S_0 = \hat{s}$, $\hat{P}$ solves problem (6). If not, go to **Step 2**.

Step 2. We must have $\lambda > 0$, otherwise we could have found an $\hat{s}$ in **Step 1**. Now we search for $\lambda > 0$ such that there exists unique $\tilde{s}$ that $\overline{\phi}_\gamma(s;\lambda) = 0$, and the "induced interim allocation"

$$\hat{P}(s) = \begin{cases} (1-s)^n & \text{if } s \in [0,\tilde{s}]/\bigcup_{i\in J}[\underline{y}_i, \overline{y}_i) \\ \frac{\int_{\underline{y}_i}^{\overline{y}_i}(1-s)^{n-1}\mathrm{d}s}{\overline{y}_i - \underline{y}_i} & \text{if } s \in [\underline{y}_i, \overline{y}_i) \\ 0 & \text{if } s \in (\tilde{s}, 1] \end{cases}$$

satisfies

$$\int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s = 0.$$

If we can find such a $(\lambda, \tilde{s})$ pair, $\hat{P}$ solves problem (6); if not, go to **Step 3**.

Step 3. There must exist an interval $[S_+, S_0] \subseteq [0,1]$ with $S_+ < S_0$ such that $\overline{\phi}(\cdot;\lambda) = 0$ on $[S_+, S_0]$, and there exists $\overline{P}$ with

$$0 \leq \overline{P} \leq \frac{\int_{S_+}^{S_0} (1-s)^{n-1}\,\mathrm{d}s}{S_0 - S_+}$$

such that

$$\hat{P}(s) = \begin{cases} (1-s)^n & \text{if } s \in [0, S_+] / \bigcup_{i \in J} [\underline{y}_i, \overline{y}_i) \\ \frac{\int_{\underline{y}_i}^{\overline{y}_i} (1-s)^{n-1} \mathrm{d}s}{\overline{y}_i - \underline{y}_i} & \text{if } s \in [\underline{y}_i, \overline{y}_i) \\ \overline{P} & \text{if } s \in (S_+, S_0] \\ 0 & \text{if } s \in (b, 1] \end{cases}$$

satisfying

$$\int_0^1 \widetilde{g}(s)\hat{P}(s)\,\mathrm{d}s = 0$$

solves problem (6).

C.2 Proof of Theorem 2

Part (a) follows from Theorem 1, and it suffices to prove Part (b). To that end, we first establish the aBRA analogue of Lemma 3.

Lemma 4. *In an aBRA, it is a weakly dominant strategy for any seller to not bid if her quality exceeds B, bid her quality when $q \in [\underline{b}_i, \overline{b}_i)$ for some $i = 1, \ldots, M$, bid $\underline{b}_i$ if $q \in [\overline{b}_{i-1}, \underline{b}_i)$ for some $i = 1, \ldots, M$, and bid B if $q \in [\overline{b}_M, B)$.*

Proof. Because the highest allowable bid is B, if a seller with $q \geq B$ ever wins, her payoff is at most zero, which is no better than not bidding regardless of what other sellers do.

Now consider a seller with $q \in [\overline{b}_M, B)$. By bidding B, there are two cases (recalling that $b_-^{\min}$ denotes the minimum bid among all other sellers):

- if $b_-^{\min} \leq \overline{b}_M$, the seller's expected payoff is 0;
- if $b_-^{\min} = B$, and k other sellers bid B, the seller's expected payoff is $\zeta(B-q)/(k+1)$.

If the seller bids $b \leq \overline{b}_M$ instead, when $b_-^{\min} \leq \overline{b}_M$, the seller's expected payoff is at most zero. When $b_-^{\min} = B$, and k other sellers bid B, the seller gets $\zeta(B + k\overline{b}_M)/(k+1) - q$;

but the payoff from bidding B is $\zeta(B-q)/(k+1)$, and

$$\zeta\frac{B-q}{k+1}-\zeta\left(\frac{B+k\bar{b}_M}{k+1}-q\right)=\frac{\zeta k}{k+1}\left(q-\bar{b}_M\right)\geq 0,$$

where the inequality holds because $q \geq \bar{b}_M$. Therefore, bidding B is a weakly dominant strategy for sellers with $q \in [\bar{b}_M, B)$.

To show that bidding the quality q is a weakly dominant strategy for a seller with quality $q \in (\underline{b}_i, \bar{b}_i)$ for some $i = 1, \ldots, M$, it suffices to compare bidding q with bidding B, as all other possible bids are covered by the proof of Lemma 3. If she bids her quality q, as described in the proof of Lemma 3, her payoff is at least zero when $b_{-}^{\min} \leq \bar{b}_M$. When $b_{-}^{\min} = B$ and k other sellers bid B, with probability $1-\zeta$ all other sellers are disqualified (i.e., no other sellers submit a legit bid), and therefore this seller gets $b_{-}^{\min} - q = B - q$; with probability ζ, this seller's payoff is given by $\zeta\left(\frac{B+k\bar{b}_M}{k+1}-q\right)$. If she bids B instead, with probability $1-\zeta$ she is disqualified and gets zero payoff, and with probability ζ her payoff is $\zeta(B-q)/(k+1)$, which is strictly worse than bidding q either way. This shows that bidding the quality q is a weakly dominant strategy for a seller with quality $q \in (\underline{b}_i, \bar{b}_i)$ for some $i = 1, \ldots, M$. An analogous argument shows that for a seller with quality $q \in [\bar{b}_{i-1}, \underline{b}_i]$ for some $i = 1, \ldots, M$, a weakly dominant strategy is bidding $\underline{b}_i$. This completes the proof. □

Proof of Theorem 2. Using Lemma 4, the proof of Theorem 1 indicates that the interim allocation probability induced by the aBRA, P_{aBRA}, equals to $\hat{P}$ defined by (11) in Proposition 2 F-almost everywhere on $[0,1]\setminus[S_+, S_0)$. For a seller with quality quantile $s \in [S_+, S_0)$, Lemma 4 implies that it is a weakly dominant strategy for her to bid B. With probability $1-\zeta$, this seller is disqualified, which means that she wins with probability 0. With probability ζ, this seller is qualified; following the same steps as in the proof of Theorem 1, we see that she wins the auction with probability $\int_{S_+}^{S_0}(1-s)^{n-1}\,\mathrm{d}s/(S_0-S_+)$. Thus, the interim allocation probability for this seller is

$$P_{aBRA}(s)=\zeta\cdot\frac{\int_{S_+}^{S_0}(1-s)^{n-1}\,\mathrm{d}s}{S_0-S_+}+(1-\zeta)\cdot 0=\overline{P},$$

where the second equality follows from the definition of the qualification rate ζ. Therefore, $P_{aBRA} = \hat{P}$ F-almost everywhere, which implies that the aBRA described in Part (b) of the theorem is indeed optimal. □

C.3 Proof of Corollary 2

We first claim that $-F/f$ is decreasing. To see this, differentiate to obtain $(-F/f)' = -1 - (-Ff'/f^2)$. When $f'(q) \le 0$, because $1-F$ is log-concave, $-Ff'/f^2 \ge (1-F)f'/f^2 \ge -1$; if instead $f'(q) > 0$, $-Ff'/f^2 \ge -1$ since F is log-concave. Thus, $(-F/f)' \le 0$. Then since $v(q) - q$ is strictly decreasing, for any $\gamma \in [0,1]$, both $h_\gamma(q)$ and $g(q)$ are strictly decreasing, and so are $\widetilde{g}$ and $\widetilde{h}_\gamma$. Therefore, the virtual surplus of the Lagrangian, $\widetilde{\phi}_\gamma(s;\lambda) = \widetilde{h}_\gamma(s) + \lambda\widetilde{g}(s)$ must be strictly decreasing for any $\lambda \ge 0$ and $\gamma \in [0,1]$. Consequently, for any $\lambda \ge 0$, there must exist a unique $\bar{S} \in [0,1]$ such that $\widetilde{\phi}(\bar{S};\lambda) = 0$. The result thus follows.

C.4 Proof of Corollary 3

If $\overline{\phi}_\gamma(0;\lambda^*) < 0$, then it is optimal not to buy from any potential seller. This is equivalent to a (trivial) BRA with the lone bid interval $\{0\}$.

Now suppose $\overline{\phi}_\gamma(0;\lambda^*) = m \ge 0$. Because both $v(q)-q$ and $-F(q)/f(q)$ are concave under our assumptions, so is $\phi_\gamma(q;\lambda^*)$ since it is a nonnegative linear combination of $v(q) - q$ and $-F(q)/f(q)$. Thus, $\phi_\gamma(q;\lambda)$ is single-peaked. Consequently, there exists $c \in [0,1]$ such that $\overline{\phi}_\gamma(\,\cdot\,;\lambda^*) = m$ on $[0,c]$ and decreasing on $[c,1]$. Then by Theorem 2, if $S_+ = S_0$, a BRA with one bid interval $[\underline{b}, S_0]$ is optimal; otherwise, an aBRA with one standard bid interval $[\underline{b}, F^{-1}(S_+)]$ and extra bid $F^{-1}(S_0)$ is optimal.

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